Financial Intermediation and Capital Reallocation

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Abstract

We develop a general equilibrium framework to quantify the importance of intermediated capital reallocation in affecting macroeconomic fluctuations and asset returns. In our model, financial intermediaries intermediate capital reallocation between low productivity firms with excess capital and high productivity firms who need credit. Because lending contracts cannot be perfectly enforced, capital misallocation lowers aggregate productivity when intermediaries are financially constrained. As a result, shocks originated from the financial sector manifest themselves as fluctuations in total factor productivity and account for most of the business cycle variations in macroeconomic quantities. Our model produces a pro-cyclical capital reallocation and is consistent with the stylized fact that the volatility of productivity are counter-cyclical at both the firm and the aggregate level. On the asset pricing side, our model matches well moments of interest rate spreads in the data and successfully generates a high and counter-cyclical equity premium.

Keywords: Financial Intermediation, Capital Reallocation, Volatility, Equity Premium

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I Introduction

The purpose of this paper is to develop a general equilibrium framework to understand qualitatively and quantitatively the extent to which shocks that affect capital reallocation can account for fluctuations in macroeconomic quantities and asset prices. In Figure 1, we plot total factor productivity (TFP) (dashed line) against measured capital misallocation, where both series are HP filtered. We follow a similar procedure as Hsieh and Klenow (2009) and measure capital misallocation by the variance of the cross-sectional distribution of log marginal product of capital within narrowly defined industries (classified by the four-digit standard industry classification code). We use a first order Taylor approximation as described in Appendix B to translate the dispersion measure into units of TFP losses.\(^1\) The two series track each other remarkably closely, and this pattern is robust to alternative measures of TFP and misallocation. We develop a model of financial intermediation and show that shocks originated from the financial sector is capable of generating most of the variations in capital misallocation and macroeconomic quantities in the data.

Our model emphasizes the role of financial intermediaries in facilitating capital reallocation among firms with heterogeneous productivity. Following Gertler and Kiyotaki (2010), we assume incomplete markets and agency frictions (limited enforcement of financial contracts). Different from Gertler and Kiyotaki (2010), we introduce heterogeneous productivity and imperfect substitution among firms’ output. These features of the model are designed to motivate the need for reallocating capital and labor across firms with different productivity and to quantify the benefit of such.

In the model, firms are subject to idiosyncratic productivity shocks. It is efficient to reallocate capital and labor from low productivity firms with excess capacity to high productivity firms who need to expand their operation upon the realization of such shocks.

\(^1\)We detail the data construction in Appendix B.
This arrangement requires high productivity firms to borrow through financial intermediaries, who finance the investment by borrowing from low productivity firms. However, the debt contract between financial intermediaries and low productivity firms are subject to limited enforcement. A worsening of the agency frictions impedes banks’ ability to borrow from cash rich firms and limit the efficiency of capital reallocation. When this happens, because resource allocation is inefficient, the cross-sectional dispersion of marginal product of capital widens and measured total factor productivity decline.

Because the empirical evidence in Figure 1 suggests that capital misallocation by itself may account for most of the fluctuations in TFP in the data, we ask whether our model with financial shocks only and without TFP shocks can generate significant fluctuations in macroeconomic quantities and asset prices. Our benchmark calibration produces macroeconomic fluctuations comparable to standard real business cycle (RBC) models with TFP shocks, but improves on the standard model along several dimensions. First, our framework provides a micro-foundation for total factor productivity shocks and generate macroeconomic fluctuations without resorting to technology regress. In our model, variations in the efficiency of capital reallocation are responsible for most of the fluctuations in output and measured total factor productivity. Our theory relies on shocks to agency frictions that affect intermediaries’s borrowing capacity and we use empirical evidence on interest rates to discipline our quantitative exercise.

Second, our model generates endogenous countercyclical volatility in macroeconomic quantities and asset prices. This is due to the inherit asymmetry in the amplification mechanism. On one hand, a worsening of financial frictions lowers total output and bank net worth, which increases the leverage of the banking sector, making the economy more vulnerable to additional shocks to the financing constraint. On the other hand, shocks that improve bank financing constraints lower leverage and reduce macroeconomic volatility. In the extreme case where banks are unconstrained, aggregate volatility is zero in the absence
of productivity shocks.

Third, our model produces a significantly higher volatility of equity returns than standard RBC models. The inability of standard RBC model in generating significant fluctuations in asset prices is well known. To match the high volatility of investment in the data, these models typically require a very low curvature of investment adjustment cost function. Low adjustment cost implies that the variation of Tobin’s Q, which equals the marginal cost of investment in equilibrium, must be low. For example, the volatility of Tobin’s Q in our model with exogenous TFP shocks but without financing frictions is about 0.5% per year. The volatility of Tobin’s Q in our model with financing constraints is about 3.5% per year. Capital is valuable in our model not only because it delivers future cash flows, but also because it may relax intermediaries’ borrowing constraints in the future. Variations in financial market frictions affect the marginal cost of financing constraints and translate into variations in the price of capital.

Our paper belongs to the literature on macroeconomic models with financial frictions. The papers that are most related to our are Gertler and Kiyotaki (2010), Brunnermeier and Sannikov (2014), and He and Krishnamurthy (2014). Our model builds directly on Gertler and Kiyotaki (2010) and extends their model in several dimensions. First, we allow heterogeneity in firms’ productivity and study the role of financial intermediary in facilitating capital reallocation in the cross section, whereas Gertler and Kiyotaki (2010) focus on how financial frictions affect intertemporal investment decisions. As a result, the amplification mechanism in our model is much stronger, because financial frictions affect current period output directly. Second, we assume that firms’ output are imperfect substitutes. This allows us to calibrate the elasticity of substitution among varieties as in Hsieh and Klenow (2009)

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Footnote:

and quantify importance of capital reallocation. Third, we develop a recursive method to solve the model to account for the occasionally binding constraints, which is the key mechanism that generates countercyclical volatility and countercyclical equity premium in our model. Gertler and Kiyotaki (2010) linearize their model around the deterministic steady-state, where financial constraints are always binding.

Several papers emphasize the importance of capital reallocation in understanding credit market frictions. For example, Eisfeldt and Rampini (2006), Shourideh and Zetlin-Jones (2012), Chari (2012), Chen and Song (2013), and Fuchs et al. (2013). Our paper falls into this category. Eisfeldt and Rampini (2006) provide empirical evidence that the amount of capital reallocation is procyclical and the benefit of capital reallocation is counter-cyclical. They also present a model where the cost of capital reallocation is correlated with TFP shocks to rationalize these facts. The purpose of our paper is to understand whether the time-varying cost of financial intermediation alone can generate sizable macroeconomic fluctuations and thus provide a micro-foundation for TFP fluctuations. In addition, we explicitly allow for a financial intermediary sector in our model and we use empirical evidence on bank loans and interest rate spreads to discipline our calibration. Finally, different from all of the above models, we study asset prices and macroeconomic quantities jointly in our model, in particular, the dynamics of macroeconomic volatilities and expected returns.

The idea that shocks may originate directly from the financial sector and affect economic activities follows Jermann and Quadrini (2012). Different from Jermann and Quadrini (2012), our paper focus on financial intermediation and capital reallocation and their connections with the macroeconomy.

Our paper is also related to the literature on asset pricing in production economies and recursive preferences.\(^3\) The endowment based long-run risks literature emphasizes the

importance of volatility shocks in understanding asset prices (for example, Bansal and Yaron (2004), Bansal et al. (2010)). However, standard real business cycle models typically produce very little amount of endogenous time variation in the volatility of macroeconomic quantities, negligible in terms of its risk premium. Our model endogenously generates counter-cyclical volatility in consumption, stochastic discount factor, and equity returns because of the inherit asymmetry of the amplification mechanism.

The rest of the paper is organized as follows. We provide a summary of some stylized facts that motivate the development of our model in Section II. We describe the model setup in Section III. In Section IV, we characterize the solution to our model in the two-period case and use this setup to discuss the intuition for the main insights of the fully dynamic model. In Section V, we develop a recursive method to construct Markov equilibria in the fully dynamic model and a numerical procedure to solve for such equilibria. We calibrate our model and evaluate its quantitative implications on macroeconomic quantities and asset prices in Section VI. Section VII concludes.

II Stylized Facts

In this section, we provide some stylized facts that motivate the development of our theoretical model. The first fact is about the business cycle properties of the total volume of intermediated loans:

1. The total volume of bank loans is procyclical. It is negatively correlated with measures of volatility and capital misallocation.

The above fact is what motivates our theory of financial intermediation and its connection with capital reallocation. We calculate the total volume of bank loans of the non-financial corporate sector in the U.S. from the Flow of Funds Table. Total bank
loans are calculated as the difference between total corporate credits and corporate bond issuance. The details of the data construction can be found in Appendix B.

We plot the annual changes in the total volume of bank loans and the GDP growth rate of the U.S. economy in Figure 2. The shaded areas indicate NBER defined recessions. It is clear that the total volume of bank loans is strongly procyclical. The correlation between the two series is 0.42 at the annual level.

In Figure 3, we plot the annual changes in the total volume of bank loans and the measured cross-sectional dispersion in the marginal product of capital from the COMPUSTAT dataset. We provide the details of the construction of the dispersion measure in Appendix B. Clearly, the innovations of the total volume of bank loans are strongly negatively correlated with our measure of capital misallocation — the correlation of the two series is −0.43 at the annual frequency. This is consistent with the key mechanism of our model: when banks are constrained, the total volume of bank loans decreases, and capital reallocation is less efficient.

We plot the annual changes in the total volume of bank loans and aggregate stock market volatility in Figure 4. Stock market volatility is calculated by aggregating realized variance of monthly returns. The correlation between the two time series is about −0.25 at the annual level. We also plot the cross-sectional dispersion of firm profit in Figure 5. It is clear that changes in the total volume of bank loans is strongly negatively correlated with both measures of volatility.

The rest of the stylized facts are well-known. We therefore do not provide detailed discussion here but refer to the relevant literature. The second fact is about the business cycle properties of capital reallocation. This is documented in Eisfeldt and Rampini (2006).

2. The amount of capital reallocation is procyclical and the cross-sectional dispersion of
marginal product of capital is countercyclical.

The third, fourth and fifth facts are about the cyclical properties of the volatility of macroeconomic quantities and asset returns and are well-known in the macroeconomics literature and the asset pricing literature, for example, Bloom (2009), Bansal et al. (2012) and Campbell et al. (2001).

3. The volatility of macroeconomic quantities, including consumption, investment, and aggregate output is countercyclical.

4. The volatility of aggregate stock market return is also countercyclical. Equity premium and interest rate spreads are countercyclical.

5. The volatility of idiosyncratic returns on the stock market is countercyclical.

In the following sections, we setup and analyze a general equilibrium model with financial intermediation and capital reallocation to provide a theoretical and quantitative framework to interpret the above facts.

III Model Setup

A Non-financial Firms

The specification of non-financial firms in our model follows the standard monopolistic competition setup in the capital misallocation literature, for example, Hsieh and Klenow (2009). There are three types of non-financial firms, intermediate goods producers, final goods producers and investment goods producers. Because non-financial firms do not make intertemporal decisions in our model, we suppress the dependence of prices and quantities on state variables in this subsection.
Final goods are produced by a representative firm on a perfectly competitive market using a continuum of intermediate inputs. We normalize the price of final goods to one and write the profit maximization problem of the final goods producer as:

$$\max \left\{ Y - \int_{[0,1]} p_j y_j d\bar{j} \right\}$$

$$Y = \left[ \int_{[0,1]} \frac{y_j^{\eta-1}}{y_j^{\frac{1}{\eta}}} d\bar{j} \right]^{\frac{1}{1-\eta}},$$  (1)

where $p_j$ and $y_j$ are the price and quantity of input $j$ produced on island $j$, respectively. The parameter $\eta$ is the elasticity of substitution among varieties. The constant return to scale technology and the fact that the final goods market is perfectly competitive imply that final goods producers earn zero profit in equilibrium. In this case, final goods producer’s demand function for input variety $j$ can be written as:

$$p_j = \left[ \frac{y_j}{Y} \right]^{-\frac{1}{\eta}}.$$  (2)

There is continuum of monopolistically competitive intermediate goods producers indexed by $j \in [0,1]$, each producing a different variety on a separate island.\(^4\) We use $j$ as the index for both the intermediate input and the island on which it is produced. The profit maximization problem for the producer on island $j$ is given by:

$$D_F (j) = \max \{ p_j y_j - MPK_j \cdot k_j - MPL \cdot l_j \}$$

subject to:

$$p_j = \left[ \frac{y_j}{Y} \right]^{-\frac{1}{\eta}},$$

$$y (j) = Aa_j k_j^{\alpha} l_j^{1-\alpha}.$$  (3)

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\(^4\)In the rest of the paper, we suppress the state space of varieties, $[0,1]$ to save notation.

\(^5\)We use the terminology “island” to emphasize that capital cannot move freely among producers of different input varieties. The details of capital market frictions is introduced in Section
Here, the production of variety $j$ requires two factors, capital $k_j$ and labor $l_j$. $A$ is the economy wide labor augmenting productivity. $a_j$ is island $j$ specific idiosyncratic technology shock, which we assume to be i.i.d. over time. $MPK_j$ is the rental price of capital on island $j$ and $MPL$ is the economy wide wage rate. Because our focus is on capital reallocation across islands with different idiosyncratic productivity shocks, we allow the rental price of capital to be island specific, but assume frictionless labor market across the whole economy. We use $DF(j)$ to denote the total profit of firm $j$, which is paid to households as dividend. We adopt a convenient normalization,

$$\int a_j^{\eta-1} dj = 1. \quad (4)$$

As will become clear later, the above condition implies that the average idiosyncratic productivity is one and total output is given by the standard Cobb-Douglas production function, $AK^\alpha N^{1-\alpha}$ in the absence of misallocation.

Finally, the representative investment goods producer produces investment goods with a constant return to scale and convex cost function. Their profit maximization problem can be written as:

$$DI = \max \left\{ qI - H(I, \bar{K}) \right\}, \quad (5)$$

where $q$ denotes the price of investment goods, $DI$ denotes the total profit of the investment goods producing firm, $I$ denotes the total amount of investment goods produced, and $\bar{K}$ denotes the total capital stock of the economy:

$$\bar{K} = \int K(j) dj. \quad (6)$$

In equation (5), $H(I, \bar{K})$ is the cost of investment, including adjustment cost. We assume
a standard quadratic adjustment cost:

\[ H(I, K) = I + \frac{1}{2} h \left( \frac{I}{K} - i^* \right)^2 K, \quad (7) \]

where \( h \) is a positive constant, and \( i^* \) is the steady-state investment-to-capital ratio.

**B Household**

There is a representative household with recursive preferences with constant risk aversion \( \gamma \) and constant IES \( \psi \). As in Gertler and Kiyotaki (2010), market is incomplete and household can only invest in a risk-free deposit account with financial intermediaries. We assume (and later verify) that household’s utility maximization problem can be written in a recursive fashion:

\[
V(Z, W) = \max_{C, B_f} \left\{ (1 - \beta) C^{1-\frac{1}{\gamma}} + \beta \left( E \left[ V(Z', W')^{1-\gamma} \mid Z \right] \right)^{\frac{1-\gamma}{1-\gamma}} \right\}
\]

\[ C + B_f = W \]

\[ W' = B_f R_f(Z) + \int D_F(i)(Z') \, di + D_I(Z') + \int D_B(i)(Z') \, di + MPL(Z') - \chi q(Z') \bar{K}'. \]

In the above maximization problem, we assume that there exist a vector of Markov state variables \( Z \), the law of motion of which will be specified later, that completely summarize the history of the economy.\(^6\) Taking the equilibrium interest rate \( R_f(Z) \), the dividend payment from intermediate goods producers, \( \{ D_F(i)(Z) \}_{i \in [0,1]} \), that from investment goods producers, \( D_I(Z) \), and that from the banks, \( \{ D_B(i)(Z) \}_{i \in [0,1]} \) as given, the household makes its optimal consumption and saving decisions given its initial wealth level, \( W \).\(^7\)

Footnotes:

\(^6\)In another words, we will focus on Markov equilibria with state space \( Z \), where \( Z \) is the set of all possible realizations of \( Z \). There is no general uniqueness and existence result that can be applied to the Markov equilibrium in our model. In Section IV, we construct such an equilibrium and our numerical result suggests such equilibria is unique.

\(^7\)We use the term financial intermediary and bank interchangeably.
includes total savings in the bank account, \( B_f R_f (Z) \), total dividends (monopolistic rents) from intermediate goods producers, \( \int D_F (i) (Z') di \), total dividend payment from banks, \( \int D_B (i) (Z') di \), and total labor income, \( MPL (Z) \). Here we assume that the household is endowed with one unit of labor in every period, which it supplies inelastically to firms.

The household owns the ultimate claims of all assets in the economy. As in Gertler and Kiyotaki (2010), we assume that the household owns the capital stock of the economy only indirectly through the bank. Therefore, household income does not include capital income directly but only dividend payment from banks. Also similar to Gertler and Kiyotaki (2010), we assume that household must involuntarily inject \( \chi \) fraction of the total value of capital stock, \( \chi q (Z') \bar{K}' \) into the banking sector in every period. This assumption ensures that the net worth of the banking sector will never be depleted in equilibrium.

C  Financial Intermediaries

There is one financial intermediary on each island.\(^8\) Financial intermediaries or banks specialize in capital reallocation.

We assume that the representative household is divided into bankers and workers, and there is perfect consumption insurance between bankers and workers within the household. Under this assumption, banks evaluate future cash flows using the "stochastic discount factor" implied by the marginal utility of the household: \(^9\)

\[
M' = \beta \left( \frac{C (Z', W')}{C (Z, W)} \right)^{-\frac{1}{\psi}} \left( \frac{V (Z', W')}{E [V (Z', W')^{1-\gamma} | Z]} \right)^{\frac{\gamma - \frac{1}{\psi}}{\gamma}}
\]

(8)

A new generation of banks enter into the economy at the end of every period after the

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\(^8\)Because financial intermediaries on each island face competitive capital markets, one should interpret our model as having a continuum of identical financial intermediaries on each island.

completion of production. As in Gertler and Kiyotaki (2010), we assume that capital and net worth moves freely across islands at the end of every period after the completion of current-period production and before the realization of next-period shocks. This is a simplifying assumption that avoids having to keep track of the cross-sectional distribution of net worth and capital as a state variable. As will become clear later, because island specific productivity shocks are i.i.d. over time, the ratio of capital and net worth must equalize across islands at the end of each period. In addition, there is an economy wide price for capital at the end of each period, which we denote \( q(Z) \).

Consider a bank who enters into a period with initial net worth \( N \). Given the end-of-period price of capital \( q(Z) \), the bank chooses the total amount of borrowing from the household, \( B_f \), amount of borrowing from peer banks, \( B_I \), and the total amount capital stock for the next period \( K' \), subject to the following budget constraint:

\[
q(Z) K' = N + B_f + B_I.
\] (9)

As in Gertler and Kiyotaki (2010), we assume incomplete market in that banks can only borrow from households on a risk-free account.\(^{10}\) In our model, the total amount of capital for the next period, \( K' \) is determined at the end of the current period before the realization of shocks of the next period. That is, we assume one period time to plan as in standard real business cycle models. However, different from the standard representative firm setup, capital can be reallocated across firms after idiosyncratic productivity shocks are realized, which we turn to next.

The market for capital reallocation opens after the realization of aggregate productivity shock \( A' \), and idiosyncratic productivity shock \( a' \). At this time, a bank has total amount

\(^{10}\)With a slight abuse of notation, we use \( B_f \) as both the amount of saving of the household and the amount of borrowing of the bank. We do so to save notation, because market clearing requires that the demand and supply of bank loans must equal.
capital $K'$ and total amount of debt obligation $R_f(Z)B_f + R_I(Z)B_I$. It has to borrow further on the interbank market in order to purchase more capital. Let $Q(Z')$ denote the price of capital on the capital reallocation market in state $Z'$. If the realization of the idiosyncratic productivity shock on the island is $a'$, and the bank purchases $RA(Z',a')$ amount of capital on the reallocation market, its total net worth at the end of the period after the repayment of household loan and interbank borrowing is:

$$N' = Q(Z',a') [K' + RA(Z',a')] - Q(Z') RA(Z',a') - R_f(Z)B_f - R_I(Z)B_I. \quad (10)$$

Here we use $Q(Z',a')$ to denote the price of one unit of capital on a island with island-specific shock $a'$ in aggregate state $Z'$. We allow $Q(Z')$ and $Q(Z',a')$ to be potentially different because limited commitment of financial contracts may prevent certain arbitrage opportunities to be eliminated. The term $Q(Z) RA(Z',a')$ is the total liability of interbank borrowing, and $R_f(Z)B_f$ is the total payment made to households.

No arbitrage within an island implies that

$$Q(Z',a') = MPK(Z',a') + (1 - \delta)q(Z'). \quad (11)$$

That is, the price of capital must equal to the sum of its current period rental price, $MPK(Z',a')$ and its value after depreciation, $(1 - \delta)q(Z')$. In a frictionless market the above condition and the fact $Q(Z',a') = Q(Z')$ for all $a'$ guarantees that the marginal product of capital must be equalized across all islands. In our model, misallocation may happen in equilibrium due to limited enforcement of financial contracts.

After the completion of production, and before the repayment of all debt obligations, banks have an opportunity to default on their debt, abscond with a fraction of their assets, \footnote{We allow $RA(Z',a')$ to be negative. In fact, market clearing implies that the sum of $RA(Z',a')$ across all banks must equal zero.}
and set up a new bank to operate on some other island. We assume that the amount of asset a bank can abscond with upon default is:

\[
\{MPK(Z', a') + \theta q(Z')\} [K' + RA(Z', a')] - \omega [Q(Z) RA(Z', a') + R_I(Z) B_I].
\] (12)

The total amount of capital on the island is \([K' + RA(Z', a')]\), where \(RA(Z', a')\) is purchasing using interbank loan. Upon default, the bank can take away all of the current period operating income, \(MPK(Z', a') [K' + RA(Z', a')]\), and sell a \(\theta\) fraction of its capital on the market. Similar to Gertler and Kiyotaki (2010), we assume that banks have a better technology to enforce contracts than households. This captured by the parameter \(\omega \in [0, 1]\). The interpretation is that in the event of default, a fraction \(\omega\) of interbank borrowing can be recovered. The case \(\omega = 0\) means banks are no better than households in enforcing contracts, and \(\omega = 1\) corresponds to the case of frictionless interbank market. Because a fraction \(\delta\) of capital depreciates every period, we assume \(\theta < 1 - \delta\). The possibility of default implies that the contracting between borrowing and lending banks must respect the following limited enforcement constraint:

\[
N' \geq \{MPK(Z', a') + \theta q(Z')\} [K' + RA(Z', a')] - \omega [Q(Z) RA(Z', a') + R_I(Z) B_I], \text{ for all } (Z', a'),
\] (13)

where \(N'\) is given by (10).

As in Gertler and Kiyotaki (2010), a fraction \(\lambda\) of bank’s net worth is forced to be liquidated and paid back to the household as dividend. This assumption is maintained to insure that banks do not save out of their financing constraints in the long run. Because bank net worth can be freely moved across islands at the end of every period, the value function and decision rules of banks must be linear in \(N\). This feature of the model greatly simplifies our analysis, because given the total net worth of the entire banking sector, the
equilibrium does not depend on the distribution of bank net worth across islands. We denote the value function of banks as $\mu(Z)$. A typical bank maximizes:

$$
\mu(Z) = \max_{B_f, K', Rf, RA(Z', a')} E [M' \{\lambda N' + (1 - \lambda) \mu(Z')\} | Z]
$$

by choosing total capital stock for the next period, $K'$, total borrowing from households, $R_f$, and a state-contingent plan for capital reallocation, $RA(Z', a')$ for all possible realizations of $(Z', a')$, subject to constraints (9), (10), and (13).

Two important distinctions between our model and Gertler and Kiyotaki (2010) are the heterogeneity in productivity across firms and the fact that the market for capital reallocation and interbank borrowing opens after productivity shocks realize and before production. In our benchmark calibration, we assume that $\theta$ follows a Markov chain with state space $[\theta_L, \theta_H]$. Increases in $\theta$ raise banks’ outside options and lower their borrowing capacity whenever the limited enforcement constraint (13) binds. We interpret shocks to $\theta$ as shocks to financing constraints and we are interested in the equilibrium mechanism through which these shocks translates into capital misallocation and fluctuations of macroeconomic quantities.

We make one more assumption on the productivity $A$. We assume $A = \bar{A}K^{1 - \alpha}$, where $\bar{A}$ is constant. This specification follows Frankel (1962) and Romer (1986) and is a parsimonious way to inject endogenous growth into the model. From an asset pricing perspective, this allows shocks to financial frictions to affect long-run growth and generate significant risk premium on equity. From a technical point of view, as we will see in Section V of the paper, this allows us to reduce one state variable in the construction of the Markov equilibrium.

**D Market Clearing**

Because market clearing conditions have to hold in every period and are therefore not intertemporal, we suppress the dependence of all variables on time or state variables in this
section to save notation.

The total amount of capital used on any island must equal the amount of capital the bank purchased in the last period plus the amount of capital purchased on the capital reallocation market:

\[ k(j) = K(j) + RA(j), \quad \forall j. \]  \hfill (14)

In addition, the total amount of capital reallocation and the total amount of interbank borrowing in the economy must be zero,

\[
\int RA(j) \, dj = 0; \quad \int B_I(j) \, dj = 0.
\]  \hfill (15)

We also assume that capital goods can be transported frictionlessly across islands at the end of each period after production. This assumption is made for tractability. Otherwise, the joint distribution of capital stock and productivity will typically be an (infinite dimensional) state variable for the construction of Markov equilibria. The above assumption requires that at the end of each period after production, the global market for the total amount of capital (prepared for production in the next period) must clear:

\[
\int K'(j) \, dj = (1 - \delta) \int K(j) \, dj + I.
\]  \hfill (16)

We assume that labor is perfectly mobile. The market clearing condition for labor is

\[
\int l(j) \, dj = 1.
\]  \hfill (17)

Market clearing for final goods requires that total consumption and cost of investment sum up to total output:

\[ C + H(I, \bar{K}) = Y, \]  \hfill (18)
where $H(I, \bar{K})$ is the adjustment cost function defined in (7), and $\bar{K}$ is the total amount of capital in the economy (equation (6)).

Because islands only differ in the realization of the idiosyncratic productivity shock $a$, we will replace the index $j$ by the realization of the productivity shock $a$ in the rest of the paper.

### E Markov Equilibrium

Let $N$ denote the total amount of net worth in the economy. The total amount of net worth in the economy in the next period is:

$$N' = (1 - \lambda) \left\{ \int Q(Z', a) [K' + RA(Z', a)] da - R_f(Z) B_f \right\} + \chi q(Z') \bar{K}'. \quad (19)$$

Integrating the budget constraint for banks on all islands, we have:

$$q(Z) \bar{K}' = N + B_f. \quad (20)$$

Equations (14), (19) and (20) together imply:

$$N' = (1 - \lambda) \left\{ \int Q(Z', a) [K' + RA(Z', a)] da - R_f(Z) [q(Z) \bar{K}' - N] \right\} + \chi q(Z') \bar{K}', \quad (21)$$

which is the law of motion of aggregate net worth.\(^{12}\)

A Markov equilibrium is a set of prices and quantities and the law of motion of the state variable $Z$ such that household maximizes utility, non-financial firms and financial intermediaries maximizes their profit and all markets clear. We provide a construction of the state variables and the Markov equilibrium in Section IV of the paper.

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\(^{12}\)Note that we use $N$ to denote both the net worth of an individual bank, and the net worth of the entire banking sector to simplify notation.
IV  A Two-Period Version of the Model

In this section, we explicitly solve a two-period version of the above model and use this simple setup to illustrate the main intuition of the fully dynamic model. We make several additional assumptions.

First, we assume that there are only two possible realizations of idiosyncratic productivity shocks, $a_H$ and $a_L$. We denote

$$\text{Prob}(a = a_H) = \pi; \quad \text{Prob}(a = a_L) = 1 - \pi.$$  \hspace{1cm} (22)

We will maintain this assumption in the fully dynamic model, where the distribution of $a_t$ is assumed to be i.i.d. over time and across islands. In this case, assumption (4) can be written as

$$\pi a_H^{1-\eta} + (1 - \pi) a_L^{1-\eta} = 1.$$  \hspace{1cm} (23)

Second, we assume that capital share $\alpha = 1$, that is, capital is the only input and the production technology is linear. We also assume that there is no adjustment cost, $h = 0$.

Third, the representative household has log preferences and the household’s maximization problem is written as:

$$\max \ln C_0 + \beta E[\ln C_1]$$

subject to :  

- $C_0 + B_f = W_0$
- $C_1 = R_f B_f + D_{F,1} + N_1$.

Here, because there are only two periods, total bank net worth in period one, $N_1$ is fully paid back to the household. Households’ budget constraint is greatly simplified because in the absence of adjustment cost, the profit of investment goods producer is zero, and under
the assumption of $\alpha = 1$, there is no labor income. In addition, there is no injection of net worth from the household to the banking sector in the two period model.

Fourth, we assume $\omega = 0$, and the outside option of a bank upon default is $\theta Q_j(\theta)$ for $j = H, L$. Here we use subscripts $j = H, L$ to denote the realization of the idiosyncratic productivity shock $a_H$ and $a_L$. We use the notation $Q_j(\theta)$ to emphasize that the equilibrium price of capital depends on the realization of the state variable $\theta$. Note that in a two period model, the price of capital in the second period before production, $Q_j = MPK_j$, for $j = H, L$, because capital is valueless at the end of the period. Under these assumptions, the bank’s maximization problem can be written as

$$\max_{B_f, K', \{RA_H(\theta), RA_L(\theta)\}} E \left[ M \{ \pi N_{H,1} + (1 - \pi) N_{L,1} \} \right]$$

$$K_1 = N_0 + B_f$$

$$N_{H,1} = Q_H(\theta) [K_1 + RA_H(\theta)] - Q(\theta) RA_H(\theta) - R_f B_f$$

$$N_{L,1} = Q_L(\theta) [K_1 + RA_L(\theta)] - Q(\theta) RA_L(\theta) - R_f B_f$$

$$N_{H,1} \geq \theta Q_H(\theta) [K_1 + RA_H(\theta)] \quad \text{for all } \theta,$$

$$N_{L,1} \geq \theta Q_L(\theta) [K_1 + RA_L(\theta)] \quad \text{for all } \theta.$$

In the above optimization problem, $M = \beta \frac{C_0}{C_1}$ is the stochastic discount factor implied by household consumption. $N_{H,1}$ is bank net worth in period one in the case $a_1 = a_H$, and $N_{L,1}$ is bank net worth in period one in the case $a_1 = a_L$. We assume that the shocks to the financing constraints, $\theta$ is distributed uniformly on $[\theta_L, \theta_H]$.

For simplicity, we also assume that there is no financial friction in period 0, and total output in this period is given by $A_0 K_0$, where $A_0$ is the aggregate productivity in the first period. It is convenient to define

$$\phi = \frac{K_1 + RA_H}{K_1 + RA_L}.$$
as the ratio of capital on high productivity islands with respect to that on low productivity islands. We denote \( \hat{\phi} = \left( \frac{a_H}{a_L} \right)^{\eta-1} \) to be the optimal ratio of \( \frac{K_H}{K_L} \) that equalizes the marginal product of capital across islands. Total output and marginal product of capital can be written as functions of \( \phi \), which we summarize in the following proposition.

**Proposition 1 (Aggregation of the Product Market)**

The total output of the economy is given by

\[
Y = Af(\phi) \bar{K}^\alpha,
\]

where the function \( f : [1, \hat{\phi}] \to [0, 1] \) is defined as:

\[
f(\phi) = \frac{\left( \frac{\bar{K}}{A} \right)^{1-\xi} \phi^{\xi} + 1 - \pi}{(\pi \phi + 1 - \pi)^\alpha \left( \frac{\bar{K}}{A} \right)^{\xi-\alpha}}.
\]

(30)

The marginal product of capital on low productivity islands, denoted \( MPK_L \) and the marginal product of capital on high productivity island, denoted \( MPK_H \) can be written as:

\[
MPK_L(A, \phi) = \alpha \left( 1 + \frac{1}{\eta} \right) \left( \frac{A}{\bar{K}} \right)^{1-\alpha} f(\phi) - \frac{\pi \phi + 1 - \pi}{\pi \phi + 1 - \pi},
\]

(31)

\[
MPK_H(A, \phi) = MPK_L(\phi) \left( \frac{\hat{\phi}}{\phi} \right)^{1-\xi},
\]

(32)

where the parameter \( \xi \in (0, 1) \) is defined as \( \xi = \frac{\alpha \eta - \alpha}{\alpha \phi - \alpha + 1} \).

**Proof.** See Appendix B.

Note that the function \( f(\phi) \) is a measure of misallocation. It is straightforward to show that \( f \) is strictly increasing with \( f(\hat{\phi}) = 1 \). The case \( \phi = \hat{\phi} \) is the first best scenario with maximum amount of capital reallocation and when \( \phi = 1 \), there is no capital reallocation.
When $\phi = \hat{\phi}$ the aggregate production function becomes the standard Cobb-Douglas case, and the marginal product of capital on all islands equalizes according to equations (31) and (32). In general, $f(\phi) < 1$ and $MPK_H(\phi) > MPK_L(\phi)$ for $\phi \in (1, \hat{\phi})$.

Because we have assumed that the technology is AK ($\alpha = 1$), equilibrium quantities are homogenous of degree one in $K_0$ and prices are homogenous of degree zero. We define $n = \frac{N_0}{K_0}$ and the equilibriums can be indexed by $n$. Define

$$n^* = \left(1 - \frac{1}{\eta}\right) \left(A_0 + 1 - \delta\right) \frac{1}{1 + \left(1 - \frac{1}{\eta}\right)}.$$

As we show in Appendix C, when $n \geq n^*$, banks have enough net worth to finance the first best allocation, and there is no misallocation in equilibrium. To simplify the analysis, we make the following assumption on the parameters values of the model.

**Assumption:** The parameter values of the model satisfy:

$$\pi > \theta_H \geq \theta_L > 1 - \frac{\xi}{\phi} - 1.$$

The deterministic version of the above simple model can be solved in closed form, which we summarize in the following proposition.

**Proposition 2** *(Capital Misallocation)*

Assume $\theta_H = \theta_L$, there exists $0 < \hat{n} < n^*$ such that

1. For all $n \geq n^*$, none of the limited enforcement constraints, (27) and (28) binds. In this case, there is no misallocation, and the marginal product of capital is equalized across all islands: $Q_H = Q_L = Q$.

2. For $n \in (\hat{n}, n^*)$, the limited enforcement constraint binds only on islands with $a = a_H$ (27) and not on islands with low productivity. In this case, islands with high productivity
shocks also have a higher marginal product of capital: \( Q_H > Q = Q_L \). In addition, \( \phi \in (1, \hat{\phi}) \), and is strictly decreasing in \( \theta \).

3. For \( n \leq \hat{n} \), the limited enforcement constraint bind on all islands. In this case, there is no capital reallocation, and \( \phi = 1 \). In addition, the market price of capital is lower than the marginal product of capital on both islands: \( Q_H > Q_L \geq Q \).

**Proof.** See Appendix.  

We state the above proposition under the assumption that \( \theta \) is constant, which allows us to provide closed form solutions to the model. The basic conclusion of the proposition is true in the more general stochastic case, although the proof becomes unnecessarily complicated.

By the above proposition, \( \hat{n} \) and \( n^* \) divide the state space into three regions. For \( n \geq n^* \), none of the financing constraints bind, and the marginal product of capital are equalized across all islands. This corresponds to the case of maximum amount of capital reallocation.

For \( \hat{n} \leq n < n^* \), the financing constraint on high productivity islands binds, but that on low productivity islands does not. In this case, \( Q_H > Q \). It is profitable for banks on high productivity islands to purchase more capital, as the marginal product of capital on high productivity islands is higher than the price of capital. However, they are not able to finance the efficient amount capital reallocation because they do not have enough net worth to collateralize inter-bank loans. On the other hand, \( Q = Q_L \) and banks on low productivity islands are not constrained. The marginal product of capital on low productivity islands equals the market price of capital. In this region, limited enforcement mainly constrains cross-sectional capital reallocation. There is partial capital reallocation in the economy.

If bank net worth is lower than \( \hat{n} \), then all banks are constrained. In this case, \( Q_H > Q_L > Q \). It is profitable to both banks to purchase more capital on the reallocation market because the marginal product of capital is higher than its market price. The high marginal product of capital in period one implies the economy should have invested more in period
zero. However, because bank net worth is extremely low in this case, they are not able to borrow enough from the households to finance investment. In this region, there is no capital reallocation, and limited commitment not only constrains the cross-sectional reallocation of capital, but also the intertemporal allocation of capital, that is, investment in period zero.

We illustrate the intuition of the above proposition using Figures 6-9, where we numerically solve the model for the stochastic case and allow $\theta_H > \theta_L$. In Figure 6, we plot investment rate, that is $I/K_0$ as a function of normalized net worth, $n = N_0/K_0$. For $n \geq n^*$, investment is at its first best level, 10%. As net worth decreases, investment rate drops and it does so sharply as net worth falls below $\hat{n}$ as all banks become constrained.

In Figure 8, we plot the total amount of capital reallocation as a function of normalized bank net worth (top panel) and the cross-sectional dispersion of log marginal product of capital as a function of normalized bank net worth (bottom panel). We plot these measure of capital reallocation as a function of $\theta$. The dashed line corresponds to the lowest value, $\theta_L$, and the dotted line corresponds to the case with the maximum amount of friction, $\theta_H$. For values of $n$ lower than $\hat{n}$, both banks are constrained and there is no capital reallocation in the worst case scenario, $\theta_H$. In this region, total amount of borrowing, $B_f$ and investment $I$ in period 0 is determined by the limited commitment constraint. Further decrease in bank net worth must be accompanied by reductions in borrowing from household, $B_f$ and therefore investment, $I$ to ensure that the limited enforcement constraint is not violated in the state of maximum financial friction, $\theta_H$. For value of $\theta < \theta_H$, partial capital reallocation is still possible, and the cross-sectional dispersion of marginal product of capital decreases with $\theta$ (bottom panel), indicating better capital reallocation.

As net worth $n$ increases above $\hat{n}$, capital reallocation improves for all possible realizations of $\theta$. The total amount of capital reallocation increases, and the cross-sectional dispersion of marginal product of capital shrinks. As $n$ reaches $\tilde{n}$, the level of capital reallocation reaches its first best level in the case with minimum financial market friction, $\theta_L$, and the
cross-sectional dispersion of marginal product of capital disappears in this case. As $n$ further increases, perfect capital reallocation becomes possible for a larger range of realizations of $\theta$, until $n$ reaches $n^*$, where all banks are unconstrained in all states, and capital reallocation is perfectly efficient.

In figure 7, we plot the level of output (top panel) and the volatility of output as a function of bank net worth and shocks to financial constraints. In the top panel, the dashed line is total output as a function of bank net worth in state $\theta_L$ with lowest financial friction. The dotted line is total output in the state with maximum financial friction, $\theta_H$. In the case $n > n^*$, output is at its first best level and does not respond to shocks in $\theta$ or changes in net worth. As shown in the bottom panel of the same figure, the volatility of output in this case is zero. As bank net worth decreases total output falls for two reasons, misallocation and under-investment. As $n$ falls below $n^*$, the limited enforcement constraint does not bind in state $\theta_L$ until $n$ hits $\bar{n}$. However, total output drops nevertheless in state $\theta_L$ relative to the first best level. Anticipating the possibility of a binding constraint in period one, banks optimally lower the amount of borrowing from households and cut investment in period zero. As a result, the total capital stock and output in period one is lower than their first best levels even when the limited enforcement constraint does not bind. The gap between the dashed line and dotted line is output loss due to capital misallocation. Note that misallocation is more sensitive to shocks in $\theta$ when net worth is low. This is the key feature of the model that generates countercyclical volatility: in bad times when bank net worth is low, leverage ratio is high, as a result, misallocation is more sensitive to shocks to financial constraints. As shown in the bottom panel of the figure, volatility of output is monotonically decreasing in bank net worth. In the two period model, the only source of volatility is shocks to $\theta$. In a dynamic model, shocks to financing constraint also affect bank net worth, creating persistence and amplification in macroeconomic fluctuations.

We plot the leverage ratio of the banking sector as a function of normalized bank net
worth in Figure 9. This figure highlights the asymmetry of the effect of shocks to financing constraints in our model: positive shocks to $\theta$ tightens banks’ financing constraint and lowers output as well as bank net worth. This increases bank leverage and make the economy more vulnerable to further shocks. As a result, our fully dynamic model features endogenously counter-cyclical volatility in macroeconomic quantities and asset prices.

V Construction of the Markov Equilibrium

Because of the financing constraints, the equilibrium does not have a planner representation. Therefore the standard construction of recursive equilibrium (for example, Stokey and Lucas (1989), Ljungqvist and Sargent (2004)) does not apply in our model. In this section, we construct a Markov equilibrium of the economy described in Section III of the paper as the fixed point of a vector of equilibrium functionals. In particular, we conjecture that a Markov equilibrium exists with state variable $\mathbf{Z} = (\phi, \theta, n)$, where $\phi$ is the current level of $K_H/K_L$, $\theta$ is the parameter of bankers’ outside options, $n$ is total net worth normalized by current period total capital stock $K$. To motivate some of details of the equilibrium construction, we first make several observations. Let $\zeta_H$ denote the Lagrangian multiplier of the constraint (13) for high productivity banks and $\zeta_L$ be the Lagrangian multiplier for the same constraint with $a = a_L$. The first order condition with respect to banks’ choice of $B_f$ imply:

$$E \left[ \tilde{M}' \left\{ 1 + \zeta_H (Z') + \zeta_L (Z') \right\} \right] R_f (Z) = \frac{1}{q(Z)} E \left[ \tilde{M}' \left\{ 1 + \zeta_H (Z') + \zeta_L (Z') \right\} Q (Z') \right] \left| Z \right| ,$$

where given the expression of $M'$ in (8), $\tilde{M}'$ is defined by:

$$\tilde{M}' = M' \left\{ \lambda + (1 - \lambda) \mu (Z') \right\} .$$
The Lagrangian multipliers must satisfy:

\[ \pi [MPK_H (\phi') - MPK_L (\phi')] = \zeta_H (Z') O (Z'), \]  
\[ (1 - \pi) [MPK_H (\phi') - MPK_L (\phi')] = \zeta_L (Z') O (Z'). \]  

where we denote

\[ O (Z') = (1 - \omega) Q (Z') - [1 - \delta - \theta'] q (Z'), \]  

and the marginal product of capital are functions of \( \phi \):

\[ MPK_L (\phi) = \alpha \left( 1 - \frac{1}{\eta} \right) \bar{A} f (\phi) \frac{\pi \phi + 1 - \pi}{\pi \phi^{1 - \xi} \phi^\xi + 1 - \pi}, \]  
\[ MPK_H (\phi) = \alpha \left( 1 - \frac{1}{\eta} \right) \bar{A} f (\phi) \frac{\pi \phi + 1 - \pi}{\pi \phi^{1 - \xi} \phi^\xi + 1 - \pi} \left( \frac{\phi}{\hat{\phi}} \right)^{1 - \xi}. \]  

Intuitively, optimality requires that banks must be indifferent between borrowing from the risk-free rate \( R_f (Z) \) and purchasing an additional capital, \( K' \). The left hand side of (33) is the marginal cost of borrowing, and the right hand side of the same equation is the marginal benefit of investing in capital. The current-period price of capital is \( q (Z) \), investing in an additional unit of capital saves the bank \( Q (Z') \) in the next period as \( Q (Z') \) is the unit price of capital in the reallocation market. In addition, it relaxes the limited enforcement constraints, the benefit of which are measured by \( \zeta_H \) and \( \zeta_L \), respectively.

The first order condition on the choice of interbank risk-free loan implies

\[ E \left[ \tilde{M} (1 + (1 - \omega) \zeta (Z')) \right] R_f (Z) = E \left[ \tilde{M} (1 + \zeta (Z')) \right] R_f (Z). \]  

27
The envelope condition implies that \( \mu(Z) \) must satisfy:

\[
\mu(Z) = E \left[ \tilde{M}' \left( 1 + \zeta_H(Z') + \zeta_L(Z') \right) \right] R_f(Z). \tag{40}
\]

Given the convexity nature of the optimization problems of banks, households and firms, the above first order conditions and the constraints are equivalent to optimality. These conditions can then be used to construct the equilibrium. We use lower case to denote equilibrium quantities normalized by total capital stock, for example, \( c = \frac{C}{K}, b_f = \frac{B_f}{K}, \) \( n = \frac{N}{K} \) stands for household consumption, household investment in the risk-free bond, and bank net worth normalized by current period capital stock. Below we describe a procedure to construct a Markov equilibrium with state space \( Z = \left[ 1, \hat{\phi} \right] \otimes [\theta_L, \theta_H] \otimes (0, \infty) \), where \( \left[ 1, \hat{\phi} \right] \) is the set of feasible values of \( \phi \), \( [\theta_L, \theta_H] \) is the set of possible values of \( \theta \) and \( (0, \infty) \) is set of possible realizations of \( n \).

We construct the Markov equilibrium by starting with an initial guess of a set of equilibrium functionals, \( \{c(Z'), v(Z'), q(Z'), \mu(Z')\}_{Z \in Z} \). Assuming the initial guess \( c(Z'), v(Z'), q(Z'), \mu(Z') \) is part of the equilibrium prices and quantities in the next period. We use the optimality conditions and the constraints to construct the policy functions of household, banks, and firms, including household consumption, price of capital, and the value function of the bank, which we denote as \( \{Tc(Z), Tv(Z), Tq(Z), T\mu(Z)\}_{Z \in Z} \). The equilibrium functional is the fixed point where \( T[c, v, q, \mu] = [c, v, q, \mu] \). We show that any fixed point of the \( T \) operator constitute a Markov equilibrium in the sense that a Markov equilibrium defined in Section II can be constructed from the equilibrium functional. This construction also provides a recursive method that can be used to solve the equilibrium numerically, which we detail in Appendix D.

Given the equilibrium functional, \( \{c(Z), v(Z), q(Z), \mu(Z)\}_{Z \in Z} \), suppose that there ex-
ists a vector of policy functions,

\[
\{ T_c(Z), T_v(Z), T_q(Z), T\mu(Z), i(Z), b_f(Z), [\zeta_H(Z, \theta), \zeta_L(Z, \theta), \phi(Z, \theta), Q(Z, \theta)]_{\theta \in [\theta_L, \theta_H]} \}_{Z \in \mathbb{Z}}
\]

such that conditions 1-8 below are met. We first define the law of motion of the state variables using the equilibrium functionals and policy functions. For each possible realization of the aggregate productivity shock, \( \theta' \), the law of motion of the state variable \( n \) is given by:

\[
n(Z, \theta') = (1 - \lambda) \alpha \left( 1 - \frac{1}{\eta} \right) \bar{A}f(\phi(Z, \theta')) - (1 - \lambda) R_f(Z) \left[ q(Z) - \frac{n}{1 - \delta + i(Z)} \right]
+ [(1 - \lambda) (1 - \delta) + \chi] q(\phi(Z, \theta'), \theta', n(Z, \theta')) ,
\]

where the interest rate \( R_f(Z) \) is constructed as follows. First use the policy \( \{ T_c(Z), i(Z) \}_{Z \in \mathbb{Z}} \) and formula (8) to construct the stochastic discount factor as a function of \( \theta' \):

\[
M(Z, \theta') = \beta [1 - \delta + i(Z)]^{-\frac{1}{\varphi}} \left[ \frac{C(\phi(Z, \theta'), \theta', n(Z, \theta'))}{T_c(Z)} \right]^{-\frac{v}{\varphi}} \left[ \frac{V(\phi(Z, \theta'), \theta', n(Z, \theta'))}{\left( E[V(\phi(Z, \theta'), \theta', n(Z, \theta'))^{1-\gamma}] \right)^{\frac{1}{1-\gamma}}} \right]^{\frac{1}{\varphi-\gamma}},
\]

and second, the interest rate is constructed from household’s first order condition:

\[
R_f(Z) = E[M(Z, \theta')]^{-1} . \tag{42}
\]

The equilibrium conditions can be summarized by:

1. Banks’ intertemporal optimality condition (33).

2. Optimality with respect to capital reallocation, conditions (34) and (35).
3. The policy $T\mu (Z)$ satisfies the envelope condition (40):

$$T\mu (Z) = E [M (Z, \theta') \{ \lambda + (1 - \lambda) \mu (Z, \phi (Z, \theta'), \theta', n (Z, \theta')) \} \{ 1 + \zeta_H (Z, \theta') + \zeta_L (Z, \theta') \} R_f (Z)] .$$

(43)

4. The limited enforcement constraints, (13), written in normalized terms:

$$\begin{align*}
(1 - \delta - \theta') q (Z') - \frac{b_f (Z) R_f (Z)}{1 - \delta + i (Z)} & \geq O (Z') \left( \frac{1 - \pi}{\pi \phi (Z, \theta') + 1 - \pi} \right) \left( \phi (Z, \theta') - 1 \right), \\
(1 - \delta - \theta') q (Z') - \frac{b_f (Z) R_f (Z)}{1 - \delta + i (Z)} & \geq -O (Z') \frac{\pi}{\pi \phi (Z, \theta') + 1 - \pi} \left( \phi (Z, \theta') - 1 \right) ,
\end{align*}$$

(44)

(45)

where we denote $Z' = (\phi (Z, \theta'), \theta', n (Z, \theta'))$, and the $O (Z')$ function is as defined in (36).

5. The resource constraint:

$$T_c (Z) + i (Z) + \frac{1}{2} h (i (Z) - i^*)^2 = \bar{A} f (\phi)$$

(46)

6. The budget constraint for banks:

$$T_q (Z) [1 - \delta + i (Z)] = n + b_f (Z)$$

(47)

7. The optimality condition for investment goods producers:

$$T_q (Z) = 1 + h (i (Z) - i^*) .$$

(48)
8. The policy $Tv(Z)$ is household continuation utility:

$$Tv(Z) = \left\{ (1 - \beta) c(Z)^{1-1/\psi} + \beta [1 - \delta + i(Z)]^{1-1/\psi} \left( E \left[ v(\phi(Z, \theta'), \theta', n(Z, \theta'))^{1-\gamma} \right] \right)^{1-1/\psi} \right\}^{1-1/\psi},$$  

(49)

Note that the above procedure defines a mapping from the space of equilibrium functionals to itself through the $T$ operator. Although the existence and uniqueness of the $T$ operator are difficult to establish in general and are beyond the scope of the current paper, whenever the fixed point does exist, as shown by the proposition below, we can construct a Markov equilibrium of our model out of the fixed point of equilibrium functionals. In practice, the $T$ operator gives us a recursive procedure that if convergent, can be used to compute the equilibrium numerically. Our numerical method is based on the $T$ operator constructed above.

**Proposition 3** (Markov Equilibria)

Suppose the $T$ operator has a fixed point in the space of bounded continuous functions, which we denote as

$$\{c^*(Z), v^*(Z), q^*(Z), \mu^*(Z)\}_{Z \in Z}$$  

(50)

We can construct a Markov equilibrium as follows:

1. The Markov state variable is $Z = (\phi, \theta, n)$. The exogenous process $\theta_t$ is Markov with state space $[\theta_L, \theta_H]$. The law of motion of $n$ is given by (34) and for each realization of $\theta'$, the law of motion of $\phi$ is given by $\phi' = \phi(Z, \theta')$, where $\phi(Z, \theta)$ is the policy function given the equilibrium functionals in (50).

2. The equilibrium prices can be constructed as follows. Interest rate $R_f(Z)$ is given by (42), and the interbank risk-free interest rate is given by (39). The equilibrium price
of capital is \( q^* (Z) \), and for \( i = H, L \),

\[
Q_j (Z) = MPK_j (\phi (Z)) + (1 - \delta) q (Z). 
\]  

(51)

3. Given the above price system, given an initial condition of the economy, \( \{ \phi_0, \theta_0, n_0, \bar{K}_0 \} \).

Suppose \( W_0 = c (Z_0) \bar{K}_0 + q (Z_0) \bar{K}_0 [1 - \delta + i (Z_0)] - n_0 \). For \( t = 0, 1, 2 \cdots \), let the total capital stock be given by:

\[
\bar{K}' = \bar{K} [1 - \delta + i (Z)],
\]

(52)

and household wealth be given by:

\[
W_t = c (Z) \bar{K}_t + q (Z_t) \bar{K}_{t+1} - n_t \bar{K}_t.
\]

(53)

The the optimal policy for households can be constructed as:

\[
C (Z_t, W_t) = c (Z_t) \bar{K}_t,
\]

\[
B_f (Z_t, W_t) = q (Z_t) \bar{K}'_{t+1} - n_t \bar{K}_t.
\]

4. The optimal policy for banks are given by:

\[
B_f (Z_t) = b_f (Z_t) \bar{K}_t
\]

\[
RA_H (Z_t) = (1 - \pi) [\phi (Z) - 1] K_t,
\]

\[
RA_L (Z_t) = -\pi [\phi (Z) - 1] K_t.
\]

In the above construction, state variables consist of the exogenous shock, \( \theta \), and the distribution of capital stock across firms, \( \phi \), and normalized bank net worth, \( n \). Note that
$W$ is a state variable in the household portfolio choice problem, and bank net worth $N$ is a state variable in banks’ problem. In general, both $W$ and $N$ should be part of the state variables in the construction of a Markov equilibrium. However, the Walras law implies that $N$ and $W$ must satisfy

$$N + W = C(Z) + q(Z) K'(Z).$$

Therefore, there is only one degree of freedom between $N$ and $W$. We choose bank net worth $N$ to be the state variable in our recursive formulation of the equilibrium.

VI Quantitative Results

In this section, we calibrate our model and evaluate its quantitative performance. Our choice of the preference parameters follows the standard long-run risks literature: we choose the risk aversion to be 10, and IES to be 2. Several parameters for the production technology are also standard in the real business cycle literature, for example, capital share and depreciate rate of capital. We choose $\pi = 0.1$ so that half of the output are produced by high productivity firms in the first best allocation. We choose the two key parameters that governs the gain from capital reallocation from the estimates of Hsieh and Klenow (2009). We choose the elasticity of substitution among varieties to be $\eta = 4$, and $a_H/a_L = 2.08$ to match the interquartile range of the estimated distribution of productivity in Hsieh and Klenow (2009). We choose the productivity parameter $\bar{\bar{A}} = 0.71$ to match the average growth in the U.S. of 1.8% per year in the post war period. We choose $\beta = 0.977$ to match an average risk-free interest rate (the three-month T Bill rate) of 0.86%. In the quantitative exercise, we use an adjustment cost similar to Gertler and Kiyotaki (2010):

$$H(I_t, I_{t-1}) = I_t + \frac{1}{2} h \left( \frac{I_t}{I_{t-1}} - (1 - \delta + i^*) \right)^2,$$
where $1 - \delta + i^*$ is the steady-state growth rate of the economy. We set $h = 0.65$.

The specification of banks’ outside option is important for the quantitative magnitude of the amplification mechanism. We choose the long-run mean of $\theta$, $\bar{\theta} = 0.35$, and $\omega = 0.35$. These choices allow us to match the spread between interbank loan and household loan in our model, $R_f(Z) - R_f(Z)$ with the average TED spread (the spread between LIBOR and U.S. treasury bills rate) of 0.64% during our sample period of 1986-2011. In addition, it implies that in steady state, 90% of total investment are financed by capital reallocation.

The amount of capital reallocation in our calibration is quite large. Empirical evidence suggests that for publicly traded firms, about 25% of total investment are financed from external funds, for example Eisfeldt and Rampini (2006). Using UK data for privately held firms, Shourideh and Zetlin-Jones (2012) find roughly 95% of investment are externally financed. We choose to match a moment that is closer to the evidence for private firms. Note that a smaller amount of capital reallocation implies that firms are more constrained in the steady-state. This implies that total output is more sensitive to capital misallocation. From this point of view, our calibration may understate the effect of capital reallocation.  

We choose the autocorrelation of $\theta$ to be 0.95, so that the equilibrium consumption dynamics are consistent with that in the long-run risks literature. We do not have much guidance in choosing the volatility of innovations in $\theta_t$, we set it to be 8% per year, this produces a volatility of total output of 2.5% per year, comparable to the volatility of total output in the data. Even if the volatility of $\theta$ is not directly observable, the volatility of the inter-bank risk-free interest rate can be used to discipline this moment. Large volatility in innovations in $\theta$ will produce excessive volatility in inter-bank lending rates. The volatility of interbank interest rate in our model is 2.08% per year, matching closely the volatility of annualized three-month LIBOR in the data, 2.16% per year.

\footnote{At the same time, because financial friction is the only friction that affects capital reallocation, our model would produce too high an interest rate on the interbank market if calibrated to match lower levels of capital reallocation. This is why we do not entertain this possibility here.}
We display the moments of macroeconomic quantities generated by our model in Table 2. These moments are comparable to standard real business cycle models in that our model produces a low volatility in consumption and a much higher volatility in investment. However, the correlation between consumption and investment is considerably lower than their data counterpart and than that in standard RBC models. Note our model does not have productivity shocks, but behave like a model with TFP shocks in terms of macroeconomic fluctuations. Most of the volatility in output are generated from misallocation. The volatility of the efficiency measure, \( f(\phi) \) is 2.05\% per year, accounting for almost all of the variations in total output, the rest of the volatility originating from variations in capital accumulation. The cross-sectional variance of measured marginal product of capital in our model is quite volatile (2.8\% per year), and is highly countercyclical: its correlation with measure \( TFP \) is \(-62\%\) at the annual level.

Financial shocks in our model generate a significant amount of fluctuations in macroeconomic quantities despite the small amount of volatility in shocks to \( \theta \). Two features are responsible for the significant impact of agency frictions on aggregate output in our model. First, the total amount of capital are predetermined as in neoclassical models; however, capital reallocation happens immediately after the realization of productivity shocks and affect current period output directly. This aspect of our model captures the idea that transfer of ownership of capital happens at a higher frequency than the formation of capital. Quantitatively, it allows shocks to translate into misallocation immediately, generating significant impact on total output. Kocherlakota (2000) points out that credit market frictions are unlikely to have a large impact on real output in models where these frictions restrict intertemporal capital accumulation. The basic reason is that investment account for about 10\% of capital and capital income account for about \( 1/3 \) of total income. Roughly speaking, a one dollar reduction in investment will translate into a decrease in next period output by about three cents, a fairly small number. Similar to us, in a model where financial constraints
affect current period output directly, Jermann and Quadrini (2012) also find credit market frictions to have a quantitatively large impact on real quantities.\footnote{In a similar model where limited enforcement affect capital reallocation in the next period, but not current period, we found that capital market frictions affect mainly intertemporal investment, and their impact on misallocation is typically negligible.}

The second reason for the significant impact of financial frictions is the equilibrium amplification mechanism. To understand this point, note that whenever $O(Z') > 0$, the limited commitment constraint on low islands does not bind, and $Q(Z') = Q_L(Z')$.\footnote{See Appendix D} In this case, the term $O(Z')$ can be written as: $(1 - \omega) MP K_L (\phi') + [\theta - \omega (1 - \delta)] q(Z')$ and constraint (44) becomes

$$
(1 - \delta - \theta') q(Z') - \frac{b_f(Z) R_f(Z)}{1 - \delta + i(Z)} \geq \left[ (1 - \omega) MP K_L (\phi') + [\theta - \omega (1 - \delta)] q(Z') \right] \frac{(1 - \pi) [\phi(Z, \theta') - 1]}{\pi \phi(Z, \theta') + 1 - \pi}.
$$

(54)

The left hand side of the above constraint is the net benefit of continuation, and the right hand side is the net benefit of default. An additional unit of capital increases the net benefit of continuation by $(1 - \delta - \theta') q(Z')$, because it increases bank net worth by $(1 - \delta) q(Z')$ and raises the value of default by $\theta' q(Z')$. The term $\frac{b_f(Z) R_f(Z)}{1 - \delta + i(Z)}$ is bankers’ debt liability normalized by the size of capital. The term $\frac{(1 - \pi) [\phi(Z, \theta') - 1]}{\pi \phi(Z, \theta') + 1 - \pi}$ is the amount of capital reallocated to high productivity islands (normalized by the size of capital stock), and $O(Z') = (1 - \omega) MP K_L (\phi') + [\theta - \omega (1 - \delta)] q(Z')$ is the marginal benefit of default per unit of capital purchased on the reallocation market. The parameter $\omega > 0$ captures the idea that banks are better than households in enforcing lending contracts.

An increase in $\theta'$ tightens banks’ borrowing constraint for two reasons. The first is a direct effect because it increases bankers’ outside option. This is captured by the fact that the left hand side of (54) is decreasing and the right hand side is increasing in $\theta'$, keep all
else constant. The second effect is an equilibrium mechanism. High values of $\theta'$ implies that banks are more constrained and are under pressure to sell their assets. In addition, the persistence of $\theta'$ implies future production will be less efficient and the marginal product of capital is likely to be low. Both forces tend to lower the value of capital $q(Z')$. This further decreases the value of continuation on the left hand side of (54).\(^{16}\) Furthermore, both effects are amplified by leverage: a small percentage change in in the term $(1 - \delta - \theta') q(Z')$ may result in a large percentage in the left hand side of (54) if bank debt is large. We choose a conservative calibration where the steady-state bank leverage in our model is about 1.67. The amplification effect in our model will be stronger if we target a bank leverage of four as in Gertler and Kiyotaki (2010).\(^{17}\) Whenever constraint (54) is binding, variations in $\theta'$ and $q(Z')$ must the absorbed by interbank borrowing, \(\frac{(1-\pi)\left[\phi(Z,\theta')\right]^{-1}}{\pi\phi(Z,\theta')}\), resulting in large variations in capital reallocation and aggregate output.

To better understand the amplification mechanism in our model, we display calibrated moments for two other models in Table 2. We consider a model with TFP shocks only (column ”TFP shocks”) and shocks to capital depreciation rates only (column ”Dep Shocks”). As in many papers that emphasize the amplification mechanism, credit market frictions amplify the impact of productivity shocks on the real economy in the model with TFP shocks only. In the model with capital depreciation shocks, we follow Gertler and Kiyotaki (2010) and inject exogenous shocks to the capital depreciation rate, $\delta_t$. The volatility of exogenous shocks and the adjustment cost parameters in both calibrations are chosen to match the volatility of consumption and investment with those in the benchmark model with

\(^{16}\)Note that decreases in $q(Z')$ may also lower banker’s outside options, which is the right-hand side of (54). However, when $\omega$ is large, the coefficient $\theta - \omega (1 - \delta)$ is small or even negative. Intuitively, upon default, bankers take all of the current period profit, $MPK_L(\phi')$, and a fraction $\theta'$ of the capital stock. If frictions on interbanking borrowing is relatively small most of the capital stock borrowed from the interbank market must be returned to the lending bank. This makes default less attractive when market valuation $q(Z')$ is high.

\(^{17}\)However, a leverage ratio of four implies an unreasonably high inter-bank interest rate in our model. We therefore chose not to target the same moment in our calibration.
θ shocks. All other parameters are the same as those in the benchmark calibration.

We note that the volatility of the efficiency of capital reallocation, f(ϕ) in both calibrations is an order of magnitude smaller than the model with financial shocks (shocks to θ). The reason for this is that these models generate very small volatility in Tobin’s q. This highlights another difficulty for credit market frictions to generate a large impact on the real economy in general equilibrium models with financial intermediaries. In most models with financial intermediaries, credit market frictions affect the real economy because changes in intermediary net worth affect their borrowing capacity. However, it is a well known difficulty that standard general equilibrium production economies produce very little variation in the price of capital (unless one is willing to entertain the possibility of large fluctuations in discount rates or extraordinarily low levels of investment volatility). As a result, although many endowment based models with financial intermediary are able to generate large financial crisis, it is not clear whether one can generalize those results and conclude a large impact on the real economy. The amplification effect in the model with TFP shocks are fairly small for exactly this reason. In the model with depreciation shocks, the amplification effect could be potentially large if one is willing to specify a large volatility of capital depreciation. However, this will have to be associated with large variations in the capital stock and counterfactually high levels of volatility in total output that comes from changes in capital stock.

We now turn to the asset pricing implications of the model, which we report in Table 3. Our model produces a low and smooth risk-free interest rate as in standard long-run risks models due to the high intertemporal elasticity of substitution. Our model improves significantly upon standard long-run risks production models in several dimensions. First, our model generates a significant risk premium of the market return, 3.63% per year without accounting for financial leverage. Part of the equity premium is due to liquidity premium, in the sense that when banks are constrained they cannot purchase more capital despite the high expected return. The liquidity premium account for roughly 0.9% of the equity premium
in our calibration. The rest of the equity premium is compensation for risks. Thanks to the persistence of the finance shocks and the linear technology, our model endogenously generates long-run risks as in Ai (2010) and Kung and Schmid (2013), and therefore features a very volatile pricing kernel as in standard long-run risks models. Second, our mode generates significant volatility on the return on equity. The volatility of the unlevered equity return is 3.59%, significantly higher than standard production economies. In addition, the level and volatility of the risk-free interbank lending rate in our model is quite in line with the data.

Note also that our model endogenously generates persistent and countercyclical volatility in aggregate consumption. In our model, both idiosyncratic and aggregate volatility are counter-cyclical. In periods with tightened financial constraints, capital reallocation is limited. This implies that most of the idiosyncratic productivity shocks must be absorbed by prices. As a result, the cross-sectional dispersion of price of capital and return on equity widens. In our model, the correlation between the cross-sectional variance of return on capital and GDP growth is $-63\%$. At the same time, our model features counter-cyclical volatility of aggregate consumption, as in standard endowment based long-run risk models. The correlation of consumption volatility and GDP growth in our simulated model is about $-13\%$ at the annual level, and increases to $-47\%$ at a five-year horizon. Variations of aggregate volatility is determined by the joint dynamics of net worth and $\theta$. Within a period for a fixed level of debt, small values $\theta$ improves capital reallocation. As $\theta$ becomes smaller, output becomes less sensitive to capital reallocation. In fact, when constraint (54) stops binding, output is independent of $\theta$. Therefore, growth and volatility is negatively correlated within a period. Over longer horizons, continued negative shocks depletes bank net worth and increases leverage. This makes the economy more vulnerable to shocks on the financial market, and increases the volatility level at lower frequencies.
VII Conclusion

We presented a general equilibrium model with financial intermediary and capital reallocation. Our model emphasize the role of financial intermediary in reallocating capital across firms with heterogeneous productivity. We show that shocks to financial frictions alone may account for a large fraction in the fluctuations of measured TFP and aggregate output. Our calibrated model is consistent with the salient features of business cycle variations in macroeconomic quantities and asset prices. It improves substantially over stand RBC models in terms of asset pricing implications.

An important next step is to infer or impute shocks to financial frictions from the data and investigate whether our model can account for the realized variations in macroeconomic quantities and asset prices once these shocks are fed into the model. One possible way is to infer financial frictions from the dispersion in the marginal product of capital in the data. The close link between the dispersion measure and TFP in Figure 1 suggests that our model hold promises. A stronger discipline may be imposed on the model if we can infer shocks to $\theta$ directly from banks’ balance sheet variables. We leave this for future research.
A Misallocation and Aggregation on the Product Market

Aggregation

We first derive an aggregation result that is similar to Hsieh and Klenow (2009) and Hopenhayn and Neumeyer (2008). In fact, the product market of our model is a special case of Hsieh and Klenow (2009) and Hopenhayn and Neumeyer (2008) without labor market distortions.

Consider the maximization problem in (3), first order conditions with respect to \( k(j) \) and \( l(j) \) imply:

\[
(1 - \alpha) \left( 1 - \frac{1}{\eta} \right) p_j y_j = MPL \cdot l_j \tag{55}
\]

\[
\alpha \left( 1 - \frac{1}{\eta} \right) p_j y_j = MPK_j \cdot k_j \tag{56}
\]

Together, the above imply:

\[
\frac{k_j}{l_j} = \frac{MPL}{MPK_j} \frac{\alpha}{1 - \alpha} \tag{57}
\]

To save notation, we denote \( A_j = Aa(j) \) in this section. Note also, total output of firm \( j \) can be written as:

\[
y_j = A_j k_j^\alpha l_j^{1-\alpha} = A_j \left[ \frac{k_j}{l_j} \right]^\alpha l_j \tag{58}
\]

\[
= A_j \left[ \frac{l_j}{k_j} \right]^{1-\alpha} k_j \tag{59}
\]

Using (57) and (58), we can write \( l_j \) as a function of \( y_j \):

\[
l_j = \frac{y_j}{A_j} \left[ \frac{\alpha MPL}{(1 - \alpha) MPK_j} \right]^{-\alpha}. \tag{60}
\]
Similarly, (57) and (59) together implies

\[ k_j = \frac{y_j}{A_j} \left[ \frac{\alpha MPL}{(1 - \alpha) MPK_j} \right]^{1-\alpha}. \]  

(61)

Using the demand function \( p_j = \left[ \frac{y_j}{Y} \right]^{\frac{1}{\eta}} \), we can replace \( y_j \) in the above equations by \( p_j^{-\eta}Y \), and integrate across all \( j \), we have:

\[ \bar{K} = \int \frac{p_j^{-\eta}}{A_j} \left[ \frac{1}{MPK_j} \right]^{-\alpha} \, dj \left[ \frac{\alpha MPL}{1 - \alpha} \right]^{-\alpha} Y, \]

(62)

\[ \bar{L} = \int \frac{p_j^{-\eta}}{A_j} \left[ \frac{1}{MPK_j} \right]^{-\alpha} \, dj \left[ \frac{\alpha MPL}{1 - \alpha} \right]^{-\alpha} Y, \]

(63)

where \( \bar{K} \) and \( \bar{L} \) stands for the total capital and total labor employed for production, respectively. Together, (62) and (63) imply

\[ Y = \frac{\bar{K}^\alpha \bar{L}^{1-\alpha}}{\left[ \int \frac{p_j^{-\eta}}{A_j} \left[ \frac{1}{MPK_j} \right]^{-\alpha} \, dj \right]^\alpha \left[ \int \frac{p_j^{-\eta}}{A_j} \left[ \frac{1}{MPK_j} \right]^{-\alpha} \, dj \right]^{1-\alpha}}. \]

(64)

We can express \( p_j \) in (64) by functions of productivity and prices. Note that (55) and (56) imply

\[ MPK_j \cdot k_j + MPL \cdot l_j = \left( 1 - \frac{1}{\eta} \right) p_j y_j. \]

(65)

Using (60) and (61), we have:

\[ MPK_j \cdot k_j + MPL \cdot l_j = \frac{y_j}{A_j} \left[ \frac{MPL}{(1 - \alpha)} \right]^{1-\alpha} \left[ \frac{MPK_j}{\alpha} \right]^\alpha. \]

(66)

Combining (65) and (66), we have:

\[ p_j = \frac{\eta}{\eta - 1} \frac{1}{A_j} \left[ \frac{MPL}{(1 - \alpha)} \right]^{1-\alpha} \left[ \frac{MPK_j}{\alpha} \right]^\alpha. \]

(67)
Note that the normalization of price we choose in (2) implies \( \int p_j dj = 1 \). Integrating (67) over \( j \), we have:

\[
\frac{\eta}{\eta - 1} \left[ \frac{MPL}{(1 - \alpha)} \right]^{1-\alpha} = \left\{ \int \frac{1}{A_j} \left[ \frac{MPK_j}{\alpha} \right]^\alpha dj \right\}^{-1}.
\]  

(68)

Together, (67) and (68) imply

\[
p_j = \frac{\frac{1}{A_j} \left[ \frac{MPK_j}{\alpha} \right]^\alpha}{\int \frac{1}{A_j} \left[ \frac{MPK_j}{\alpha} \right]^\alpha dj}.
\]  

(69)

Replacing \( p_j \) in equation (64) with (69), and using \( A_j = A^{1-\alpha} a(j) \), we can write \( Y = TFP \bar{K}^\alpha \bar{L}^{1-\alpha} \), where

\[
TFP = A^{-\eta} \left\{ \int \left( \frac{a_i}{MPK_j} \right)^{\eta-1} \frac{a_i^{\eta-1+\alpha}}{\eta-1+\alpha} \right\} \alpha.
\]  

(70)

Under the assumption (4), it is straightforward to show that \( TFP = A \) is \( MPK_j = MPK \) for all \( j \). We define

\[
EF = \frac{\left\{ \int \left( \frac{a_i}{MPK_j} \right)^{\eta-1} \frac{a_i^{\eta-1+\alpha}}{\eta-1+\alpha} \right\} \alpha}{\int \left( \frac{a_i}{MPK_j} \right)^{\eta-1} \frac{1}{MPK_j} di}.
\]  

(71)

to be the efficiency measure of capital reallocation. Under the assumption \( \ln \alpha_j \) and \( \ln MPK_j \) are jointly normally distributed, we can show that

\[
\ln EF = -\frac{1}{2} \left[ \alpha (\eta - 1) + 1 \right] \alpha \sigma^2,
\]  

(72)

where \( \sigma^2 \) is the cross-sectional variance of marginal product of capital. Note also, (72) is approximately true for arbitrary distributions as long as the deviation of \( \ln \alpha_j \) and \( \ln MPK_j \) from their mean is small. Therefore, (72) can be viewed as a first order Taylor approximation that maps the cross-sectional variance of marginal product of capital into TFP losses due to
misallocation.

Proof of Proposition 1

In the special case where $a_j$ takes on only two values, $a_H$ and $a_L$ as in (22), we define $\phi = \frac{K_H}{K_L}$ to be the ratio of capital employed on islands with high productivity shock with respect to that employed on islands with low productivity shock, as in (29). Note that

$$MPK_j = \alpha A a_j \left( \frac{l_j}{k_j} \right)^{1-\alpha} ; \quad MPL = (1 - a) A a_j \left( \frac{k_j}{l_j} \right)^{\alpha}.$$ 

Note that because labor market is perfectly mobile, $MPL$ must equalize across all islands. Using the labor market clearing condition, (17) and assumption (23), we can prove conditions (31) and (32). Using there conditions to replace $MPK_j$ in (71), the efficiency measure (71) can be written as (30). This completes the proof of Proposition 1.

B Data Construction

B.1 Misallocation and TFP

In Figure 1, we plot the measure of capital misallocation and total factor productivity. We measure the cross-sectional dispersion of TFPR following Hsieh and Klenow (2009). In the context of our model, equation (56) implies

$$MPK_j = \alpha \left( 1 - \frac{1}{\eta_j} \right) \frac{p_j y_j}{k_j}.$$

Following Chen and Song (2013), we measure $MPK_j$ by the ratio of Operating Income before Depreciation (OIBDP) to one-year-lag net Plant, Property and Equipment (PPENT). As in Hsieh and Klenow (2009), we focus on the manufacturing sector and compute the cross-sectional dispersion measure within narrowly defined industries (as classified by the 4-digit
standard industry classification code). Specifically, for firm $j$ in industry $i$, we compute

$$\frac{MPK_{i,j}}{MPK_i} = \frac{\alpha \left(1 - \frac{1}{\eta} \right) \frac{p_{i,j} y_{i,j}}{k_{i,j}}}{\alpha \left(1 - \frac{1}{\eta} \right) \frac{p_{j} y_{j}}{k_j}} = \frac{p_{i,j} y_{i,j}}{p_{j} y_{j}},$$

where $\frac{p_{j} y_{j}}{k_j}$ is measured at the industry level. We then compute the variance of $\frac{MPK_{i,j}}{MPK_i}$ for each year. This is our empirical measure of $\sigma^2$ in equation (72). We use the first order approximation in (72) to construct the time series of the misallocation measure, which is the solid line in Figure 1. The measure of total factor productivity is directly taken from the published TFP series on the Federal Reserve Bank of St Louis website. Both series are HP filtered.

B.2 Total Volume of Bank Loans

We measure the total volume of bank loans of non-financial corporate sector through the aggregate balance sheet of nonfinancial corporate business (Table B.102) as reported in the U.S. Flow of Funds Table. In particular, the bank loan is calculated as the difference between total credit market liability (Line 23) and corporate bond (Line 26). Under this construction, bank loans consist of the following credit market liability items: commercial paper (Line 24), municipal securities (Line 25), depository institution loans (Line 27), other loans and advances (Line 28) and mortages (Line 29).

C The Two-Period Model

Consider the bank’s optimization problem. Let $\zeta_H$ and $\zeta_L$ be the Lagrangian multipliers of constraints (27) and (28). The first order conditions of the optimization problem imply:

$$E \left[ M (1 + \zeta_H + \zeta_L) Q \right] = E \left[ M (1 + \zeta_H + \zeta_L) \right] R_f. \quad (73)$$
\[ \zeta_H = \frac{\pi (Q_H - Q)}{Q - (1 - \theta) Q_H}; \quad \zeta_L = \frac{(1 - \pi) (Q_L - Q)}{Q - (1 - \theta) Q_L}. \]  

(74)

Dividing both sides of banks’ budget constraint by \( K_0 \), we have:

\[ 1 - \delta + i = b_f + n, \]  

(75)

where we denote \( n = \frac{N_0}{K_0} \) and \( b_f = \frac{B_f}{K_0} \). Also, using the market clearing condition (15), \( RA_H = (1 - \pi) (\phi - 1) K_1 \) and \( RA_L = -\pi (\phi - 1) K_1 \). Dividing both sides by \( K_1 \), we can rewrite (27) and (28) as:

\[
\begin{align*}
(1 - \theta) Q_H - \frac{R_f b_f}{1 - \delta + i} & \geq \ [Q - (1 - \theta) Q_H] \frac{(1 - \pi) (\phi - 1)}{\pi \phi + 1 - \pi}, \\
(1 - \theta) Q_L - \frac{R_f b_f}{1 - \delta + i} & \geq -[Q - (1 - \theta) Q_L] \frac{\pi (\phi - 1)}{\pi \phi + 1 - \pi},
\end{align*}
\]

(76)  

(77)

In addition, \( \zeta_H > 0 \) implies that (76) must hold with equality, and \( \zeta_L > 0 \) implies that (77) must hold with equality. Because of log preference, the stochastic discount factor \( M \) can be written as:

\[ M = \frac{\beta (A_0 - i)}{A_f (\phi) (1 - \delta + i)}. \]  

(78)

Finally, the household first order condition implies

\[ R_f = \frac{1}{E[M]} = \frac{A (1 - \delta + i)}{\beta (A_0 - i) E \left[ \frac{1}{T(\phi)} \right]}. \]  

(79)

Together, equations (73)-(79) give us eight equations to solve for the eight equilibrium quantities, \( \zeta_H, \zeta_L, Q, \phi, R_f, M, b_f, \) and \( i \) as a function of \( n \).\(^{18}\) Conversely, because the optimization problem of households, banks and firms are all standard convex programing problems, any solution to the system of equations (73)-(79) can be used to construct an

\(^{18}\)Strictly speaking, in the case where \( \theta \) is uniformly distributed, (74), (76), and (77) are four equations for each possible realization of \( \theta \). Also, the equilibrium quantities, \( \zeta_H, \zeta_L, Q, \phi \) are all functions of \( \theta \).
equilibrium in the two-period model.

To prove Proposition 2, note that if $n \geq n^*$, then the first best allocation,

$$i^* = \left(1 - \frac{1}{\eta}\right) A_0 - \left(1 - \delta\right); \quad \phi(\theta) = \hat{\phi};$$

$$Q_H(\theta) = Q_L(\theta) = Q(\theta) = \left(1 - \frac{1}{\eta}\right) A$$

constitute a solution to the equilibrium system with $\zeta_h = \zeta_L = 0$. One can verify that in this case both inequalities (76) and (77) are satisfied.

To construct equilibrium quantities for $n < n^*$, we first prove the following claim.

**Claim 1** $Q(\theta) \leq Q_j(\theta)$ for $j = H, L$. In addition, $Q(\theta) < Q_H(\theta)$ implies that constraint (76) must bind and $Q(\theta) < Q_H(\theta)$ implies that constraint (77) must bind.

**Proof.** If $Q(\theta) > Q_H(\theta)$, then the bank can choose $RA(\theta)$ to be arbitrarily large to increase profit unboundedly. This cannot be consistent with any equilibrium. If $Q(\theta) < Q_j(\theta)$, then equation (74) implies that $\zeta_j(\theta)$ must be positive, and therefore the corresponding constraint must be binding. ■

To prove Proposition 2, note that in the case where $\theta_H = \theta_L$, all equilibrium quantities and prices are deterministic and the equilibrium conditions can be reduced to:

$$\left(1 - \frac{1}{\eta}\right) \beta (A_0 - i) \frac{\pi \phi + 1 - \pi}{\pi \hat{\phi}^{1-\xi} \phi^\xi + 1 - \pi} \chi = 1,$$

$$\left(1 - \frac{1}{\eta}\right) \frac{(1 - \theta) \phi^{1-\xi} \phi^\xi - \chi (1 - \pi)(\phi - 1)}{\pi \hat{\phi}^{1-\xi} \phi^\xi + 1 - \pi} = \frac{1 - \delta + i - n}{\beta (A_0 - i)},$$

$$\left(1 - \frac{1}{\eta}\right) \frac{(1 - \theta) + \chi \pi (\phi - 1)}{\pi \hat{\phi}^{1-\xi} \phi^\xi + 1 - \pi} = \frac{1 - \delta + i - n}{\beta (A_0 - i)}.$$

Let $\hat{n}$ be the solution to the above system with $\chi = 1$ (therefore, we have three unknowns,
ϕ, i, and ˆn). One can show that if \( n \in (\hat{n}, n^*) \), equation (80) and (81) have a solution for \( \phi \) and \( i \) under the condition \( \chi = 1 \). This solution can be used to construct the equilibrium price:

\[
Q_H = MPK_H(\phi); Q = Q_L = MPK_L(\phi),
\]

where \( MPK_H(\phi) \) and \( MPK_L(\phi) \) are given by (31) and (32). It is straightforward to verify in this case \( \zeta_H > 0 \) and \( \zeta_L = 0 \) and all equilibrium conditions are satisfied. In the case \( n \in (0, \hat{n}) \), equations (80)-(82) have a solution in terms of \( \phi, i \) and \( \chi \) with \( \chi < 1 \). The equilibrium prices can be constructed as:

\[
Q_H = MPK_H(\phi); Q_L = MPK_L(\phi); Q = \chi Q_L.
\]

One can verify that in this case \( \zeta_H, \zeta_L > 0 \) and both constraints (76) and (77) are binding.

**D Computation Details**

We provide details of the numerical method that we use to compute the Markov equilibrium in the paper. We first need to make some simplifications of the equilibrium conditions.

**Claim 2** Assume \( O(Z') > 0 \) for all \( Z' \), where the function \( O(Z') \) is defined in (36), then constraint (45) does not bind and \( Q(Z') = Q_L(Z) \) for all \( Z' \).

**Proof.** Note the if \( O(Z') > 0 \) then inequality (44) implies (45). In this case, \( \zeta_L(Z') = 0 \) and \( Q(Z') = Q_L(Z) \) for all \( Z' \) by equation (35). 

The above claim reduces the number of policy functions in the equilibrium construct. We use the following procedure to numerically solve the model.

1. Start from an initial guess of the equilibrium functional, \( \{c^n(Z), v^n(Z), q^n(Z), \mu^n(Z)\} \) with \( n = 0 \).
2. Start from an initial guess of the policy functions associated with the equilibrium functional with \( m = 0 \):

\[
\begin{align*}
&Tc^{n,m} (Z), Tv^{n,m} (Z), Tq^{n,m} (Z), T\mu^{n,m} (Z), i^{n,m} (Z), b_j^{n,m} (Z), [\zeta_H^{n,m} (Z, \theta), \zeta_L^{n,m} (Z, \theta), \phi^{n,m} (Z, \theta), \xi^{n,m} (Z, \theta), \eta^{n,m} (Z, \theta)], \\
&\theta \in [\theta_L, \theta_H]
\end{align*}
\]

3. Use the policy function to construct the law of motion of the state variables according to (41). Using conditions 1-8 to construct policy functions given the law of motion of state variables. Update the policy functions and set \( m = m + 1 \).

4. Iterate on step 2-3 until the policy functions converge. That is, the policy functions that are used to construct the law of motion of the state variables also solves the equilibrium conditions. The new policy functions are denoted as

\[
\begin{align*}
&Tc^n (Z), Tv^n (Z), Tq^n (Z), T\mu^n (Z), i^n (Z), b_j^n (Z), [\zeta_H^n (Z, \theta), \zeta_L^n (Z, \theta), \phi^n (Z, \theta), Q^n (Z, \theta)]]_{\theta \in [\theta_L, \theta_H]}
\end{align*}
\]

5. Set

\[
\begin{align*}
c^{n+1} (Z) &= Tc^n (Z) \\
v^{n+1} (Z) &= Tv^n (Z) \\
q^{n+1} (Z) &= Tq^n (Z) \\
\mu^{n+1} (Z) &= T\mu^n (Z)
\end{align*}
\]

and update the equilibrium functional. Iterate step 1-5 until convergence.

The above procedure, if convergent, allows us to construct the Markov equilibrium as the fixed point of the operator described in Section V.
Figure 1: Business Cycle Variations of TFP and Misallocation

Figure 1 plots the time series of total factor productivity (dashed line) in the U.S. and the cross-sectional dispersion of marginal product of capital (solid line) in the period 1963-2012. The construction of the misallocation measure follows Hsieh and Klenow (2009). We provide the details of the construction in Appendix B. We use the first order Taylor expansion in equation (72) to translate the misallocation measure into TFP losses. Both series are HP filtered.
Figure 2: Business Cycle Variations of the Total Volume of Bank Loan

Figure 2 plots the business cycle variations of the total volume of bank loans for all non-financial firms in the US corporate sector. The solid line is the changes in the total volume of bank loans and the dashed line is GDP growth. Shaded areas stand for NBER classified recessions.
Figure 3: Total Volume of Bank Loan and Capital Misallocation

Figure 3 plots the net increases in the total volume of bank loan and our measure of capital misallocation constructed from COMPUSTAT firms during the period 1958-2012.
Figure 4: Total Volume of Bank Loan and Aggregate Volatility

Figure 4 plots the net increases in the total volume of bank loan and stock market volatility in the U.S. during the period 1958-2012.
Figure 5 plots the net increases in the total volume of bank loan and the cross-sectional dispersion of firm profit for COMPUSTAT firms.
Figure 6: Investment and Bank Net Worth

Figure 6 plots the total investment (normalized by capital stock) as a function of bank net worth in the two-period model.
Figure 7: Total Output and Bank Net Worth

Figure 7 plots the conditional moments of output as a function of bank net worth in the two period model. The top panel is the level of total output as a function of bank net worth for the low friction case (dashed line) and that for the high friction case (dotted line). The bottom panel plot the conditional volatility of total output as a function of bank net worth.
Figure 8 plots the amount of capital reallocation (top panel) and the benefit of capital reallocation measured by the cross-sectional dispersion of the marginal product of capital (bottom panel) as functions of bank net worth. The dashed line corresponds to case of low financial market friction, and the dotted line is for the case with high financial market friction.
Figure 9: **Leverage and Bank Net Worth**

Figure 9 plots the leverage of the banking sector as a function of bank net worth.
Table 1 lists the parameter values we use for the calibration of our model.

<table>
<thead>
<tr>
<th>Preference Parameters</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>discount rate</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>risk aversion</td>
</tr>
<tr>
<td>$\psi$</td>
<td>IES</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Technology Parameters</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>capital share</td>
</tr>
<tr>
<td>$\eta/(\eta-1)$</td>
<td>markup</td>
</tr>
<tr>
<td>$a_H/a_L$</td>
<td>ratio of productivity</td>
</tr>
<tr>
<td>$\delta$</td>
<td>capital depreciation</td>
</tr>
<tr>
<td>$h$</td>
<td>adjustment cost</td>
</tr>
<tr>
<td>$\pi$</td>
<td>probability of high productivity</td>
</tr>
<tr>
<td>$\overline{\Lambda}$</td>
<td>aggregate productivity</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Parameters of Financial Frictions</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$</td>
<td>probability of bank exit</td>
</tr>
<tr>
<td>$\chi$</td>
<td>Equity injection to banks</td>
</tr>
<tr>
<td>$\bar{\theta}$</td>
<td>Fraction of asset divertable</td>
</tr>
<tr>
<td>$\sigma_\theta$</td>
<td>volatility of $\theta$</td>
</tr>
<tr>
<td>$\rho_\theta$</td>
<td>autocorrelation of $\theta$</td>
</tr>
<tr>
<td>$\omega$</td>
<td>inter-bank friction</td>
</tr>
</tbody>
</table>
### Table 2: Business Cycle Statistics

<table>
<thead>
<tr>
<th>Moments</th>
<th>Data</th>
<th>Benchmark Model</th>
<th>TFP Shocks</th>
<th>Dep. Shocks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average GDP Growth</td>
<td>1.8%</td>
<td>1.8%</td>
<td>1.8%</td>
<td>1.8%</td>
</tr>
<tr>
<td>Volatility of GDP Growth</td>
<td>3.49%</td>
<td>2.53%</td>
<td>2.65%</td>
<td>2.43%</td>
</tr>
<tr>
<td>Volatility of Consumption Growth</td>
<td>2.53%</td>
<td>2.05%</td>
<td>2.24%</td>
<td>2.17%</td>
</tr>
<tr>
<td>Volatility of Investment relative to volatility of consumption</td>
<td>5.34</td>
<td>2.13</td>
<td>2.14</td>
<td>2.13</td>
</tr>
<tr>
<td>Correlation of Consumption and Investment</td>
<td>39%</td>
<td>9.86%</td>
<td>72.97%</td>
<td>92.67%</td>
</tr>
<tr>
<td>Autocorrelation of consumption</td>
<td>49%</td>
<td>42%</td>
<td>4.62%</td>
<td>85.68%</td>
</tr>
<tr>
<td>Volatility of efficiency of capital reallocation</td>
<td>—</td>
<td>2.05%</td>
<td>0.16%</td>
<td>0.28%</td>
</tr>
<tr>
<td>Volatility of dispersion of MPK</td>
<td>17.09%</td>
<td>2.8%</td>
<td>0.2%</td>
<td>0.2%</td>
</tr>
<tr>
<td>Average Capital Reallocation/Total Investment</td>
<td>—</td>
<td>90%</td>
<td>95%</td>
<td>93%</td>
</tr>
<tr>
<td>Volatility of Capital Reallocation/Investment</td>
<td>—</td>
<td>85%</td>
<td>11%</td>
<td>14%</td>
</tr>
<tr>
<td>Correlation of GDP and Dispersion of MPK</td>
<td>−14.08%</td>
<td>−62.34%</td>
<td>−9.93%</td>
<td>−40.20%</td>
</tr>
<tr>
<td>Volatility of Tobin Q</td>
<td>—</td>
<td>3.50%</td>
<td>0.53</td>
<td>0.43</td>
</tr>
</tbody>
</table>

Table 2 documents the moments of macroeconomic quantities in U.S. data (1930-2009) and those generated by our benchmark model (column "Benchmark Model"), the model with productivity shocks only (TFP shocks), and the model with capital depreciation shocks (Dep Shocks).
Table 3: Asset Pricing Moments

<table>
<thead>
<tr>
<th>Moments</th>
<th>Data</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equity Premium</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E \left[ \ln R_M - R_f \right]$</td>
<td>5.71%*</td>
<td>3.63%</td>
</tr>
<tr>
<td>Volatility of Return on Capital</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma \left[ \ln R_M \right]$</td>
<td>19.79%*</td>
<td>3.59%</td>
</tr>
<tr>
<td>Risk-free Interest Rate</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E \left[ \ln R_f \right]$</td>
<td>0.86%</td>
<td>1.10%</td>
</tr>
<tr>
<td>Volatility of Risk-free Rate</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma \left[ \ln R_f \right]$</td>
<td>0.97%</td>
<td>1.20%</td>
</tr>
<tr>
<td>Interest Rate Spread</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E \left[ \ln R_I - R_f \right]$</td>
<td>0.64%</td>
<td>0.88%</td>
</tr>
<tr>
<td>Volatility of Interbank Rate</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma \left[ \ln R_I \right]$</td>
<td>2.16%</td>
<td>2.08%</td>
</tr>
<tr>
<td>Volatility of SDF</td>
<td></td>
<td>93.53%</td>
</tr>
</tbody>
</table>

Table 3 documents the asset pricing moments in the data and those generated by our benchmark model. Note that the equity premium and volatility of return in the data are not directly comparable to the moments generated from our model, because our model does not account for financial leverage.
References


