The Impact of Liquidity Costs, Estimation Error, and Uncertainty Aversion on Real Estate Asset Allocation and Portfolio Performance

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Abstract

This paper adapts recently developed analytical solutions for optimal asset allocation under estimation error and uncertainty aversion to also consider the likelihood of incurring liquidation costs. Given these adapted solutions, the paper determines optimal portfolio allocations across asset classes that include large- and small-cap equity, bonds, and commercial real estate. With this analysis the paper documents the impact on optimal portfolio weight for commercial real estate associated with separately and jointly considering liquidity costs, estimation error, and uncertainty aversion. The paper then examines how these effects on portfolio composition translate into ex-post performance. Although the considerations of both liquidity costs and estimation error have an impact on both the optimal weight on commercial real estate and ex-post portfolio performance, by far the biggest impact (associated with economically significant improvements in ex-post performance) comes from considering uncertainty aversion.

Key words: liquidity risk, portfolio allocation, commercial real estate.
JEL codes: R33, G11.

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I. Introduction

Numerous studies show that direct investment in commercial real estate significantly improves the expected performance of portfolios consisting of securities representing traditional asset classes such as large- and small-cap equity, government and corporate debt, and cash (e.g., see Sirmans and Worzala (2003), Bond et al. (2007a), and Fugazza, Guidolin and Nicodano (2007)). These studies, however, typically find allocations to commercial real estate that are extreme relative to those found in professionally managed portfolios. Studies that examine asset allocation (mostly excluding commercial real estate) typically find that such extreme allocations, while improving expected performance, typically generate poor ex-post performance. This paper reexamines the optimal allocation to commercial real estate and its contribution to ex-post portfolio performance. The goal is to quantify the impact of a set of potential risks and costs that standard allocation approaches may not adequately capture but are likely to be more significant for commercial real estate than for traditional asset classes.

Specifically, this paper examines four potential sources of risks and/or costs that may be underestimated for direct investment in commercial real estate. First, to measure returns to commercial real estate, we use market transactions data rather than appraisal data (as is more common in real estate studies). Appraisal data for commercial properties typically lags market prices and are smoothed; ex-ante estimates of real estate return

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1 A review of the literature regularly finds recommended weights on commercial real estate greater than ten percent and often more than 20 percent (see Sirmans and Worzala (2005)). (Not all studies recommend such high portfolio allocations, see for instance Kallberg et al (1996).) Yet, recent data on UK institutional investors suggests an average portfolio holding for real estate of eight percent, with small funds holding as little as three percent (Bond et al (2007a)).
covariances with the returns of other asset classes may be too small, and, thus, may under estimate the true risk in commercial real estate (see Geltner (1989, 1991)), potentially resulting in exaggerated allocations to real estate. In addition, ex-post returns based on smoothed appraisal data may also under represent the true volatility/covariance in realized returns, leading to exaggerated ex-post performance. Rather than use appraisal data, this paper uses the relatively long time-series of real estate values based on market transactions as captured in the Transactions-based Index (TBI) by Fisher et al (2007).

Second, the paper examines the impact of liquidity. The literature on liquidity and asset pricing identifies two avenues through which liquidity costs may effect asset allocation and expected returns: the level of expected liquidity costs (Amihud and Mendelson (1986)) and the variance/covariance of a security’s liquidity costs with its own raw returns and the raw returns and liquidity costs of other securities/asset classes (see Chordia et al. (2000, 2001), Hasbrouck and Seppi (2001), Huberman and Halka (1999), Jones (2001), Bekaert et al. (2003), Pastor and Stambaugh (2003), and Acharya and Pedersen (2005)). To measure liquidity for commercial real estate, some studies (Lin and Vandell (2007), Bond et al. (2007), and Cheng, Lin and Liu (2010)) use “time-on-the-market.” In these studies, greater time-on-the-market implies that the ultimate liquidation price is more correlated with market movements and, thus, is riskier. Although an important contribution to the literature, the time-on-the-market approach can be problematic for our application since reliable time-on-the-market data for a

\[2\] Partly in response to data deficiencies (particularly for the commercial real estate market), very few attempts have been made to incorporate liquidity into portfolio choice when the assets under consideration include real estate. Notable exceptions include Lin and Vandell (2007), Bond et al (2007b), Cheng, Lin and Liu (2010), and Anglin and Guo (2010).
sufficiently lengthy sample period and for a sufficiently representative cross-section of commercial real estate properties are often not available. As an alternative to time-on-the-market liquidity measures, this paper again uses information contained in the TBI (and some of its components) to estimate the statistical relationship between returns and the level and variability of liquidity costs for commercial real estate.

Third, the paper considers the impact of estimation risk (i.e., the fact that the inputs to the allocation choice problem are estimated with error). We do this because it is often the case that asset-allocation approaches based on sample estimates produce both extreme allocations across asset classes and poor ex-post performance. That is, the allocations seem to be overly sensitive to noise in the estimates. In many cases, the allocations produced by these approaches are adjusted on an \textit{ad hoc} basis. In contrast, this paper examines optimal allocation (and the resulting ex-post performance) using the Bayes-Stein approach, which determines optimal allocation given the amount of statistical uncertainty in the estimated inputs. Using these methods, the paper is better able to document the incremental contribution of commercial real estate to ex-post portfolio performance by avoiding allocations that are (1) overly sensitive to noise in the estimated moments and (2) adjusted in an \textit{ad hoc} manner.

Fourth, given the possibility of parameter uncertainty (say due to estimation error), the paper also considers the impact of investors being averse to Knightian uncertainty (see Knight (1927)). A person faces risk when the outcomes are random and the probability of each distinct outcome is known with certainty. In contrast, a person faces uncertainty (also referred to as ambiguity in the literature) when the outcomes are
random and the probability of each distinct outcome is not known with certainty. If agents’ priors suggest some distribution over the possible distributions of probabilities, then agents face Knightian uncertainty, but can assign unconditional probabilities to each possible outcome. Agents that satisfy the set of axioms that imply von Neumann-Morgenstern expected utility (Savage (1954)) are averse to risk, but are not averse to the uncertainty. However, if agents do not satisfy the independence axiom (used to justify von Neumann-Morgenstern expected utility), Gilboa and Schmeidler (1989) show that agents can be averse to uncertainty (in addition to risk) and that there exists a set of subjective probabilities that can represent an uncertainty-averse agent’s preferences.

Some applications of uncertainty aversion to single asset economies developed in the economics and finance literature (e.g., Dow and Werlang (1992a, 1992b), and Epstein and Wang (1994)) produce results that suggest uncertainty aversion might explain why actual allocations to real estate are significantly lower than those implied by standard

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3 For example, consider a gamble based on the toss of a coin; the outcome is either $1 if “heads” or -$1 if “tails,” where the events “heads” and “tails” are well-defined and exhaustive. Uncertainty occurs when the probabilities of “heads” and “tails” are determined by a random draw from a distribution that itself is drawn from an urn (of distributions). Uncertainty exists even when the distribution over the types of coins in the urn is known with certainty (e.g., the urn has one two-tailed coin and one two-headed coin). In this case, the probability of receiving a two-headed coin is 50% and, if the two-headed coin is drawn, the probability of “heads” is 1 and the probability of “tails” is 0; if, however, the two-tailed coin is drawn (which happens 50% of the time), the probability of “heads” is 0 and the probability of “tails” is 1. Thus, the probability of “heads” (and “tails”) is random (either 0 or 1 with 50 percent probability for each); unconditionally, however, the probability of “heads” is .5*1 + .5*0 = .5 and the probability of “tails” is .5*0 + .5*1 = .5. Thus, even though the unconditional probability of each outcome is the same as that which occurs when it is known for certain that the coin flipped is a fair two-sided coin (i.e., they have the same risk), the above example contains uncertainty. Uncertainty is also present when the characteristics of the urn are not known (i.e., the number of two-headed, two-tailed, and two-sided coins in the urn cannot be verified prior to the gamble). When the distribution of distributions is unknown, one could model the agent as determining the risk of such a gamble by integrating over all the possible distributions of distributions using some prior on the distribution of distributions. The issue is how the agent feels about such a gamble relative to one without ambiguity but with the same risk.
(risk-averse-only) allocations methods. In particular, this literature has shown that uncertainty aversion can lead individuals to take more muted actions as a function of price (or expected return).

In a portfolio context, cross-sectional differences in degree of uncertainty across asset classes will determine the optimal allocation; a priori, it is unknown how uncertainty aversion will affect the optimal allocation to real estate when real estate is pitted against other asset classes. Furthermore, even if the allocations to real estate are muted relative to those for other asset classes, it is also unclear a priori how such allocations will translate into ex-post portfolio performance.

To consider the impact of liquidity, estimation risk, and uncertainty aversion, the paper adapts recently developed analytical solutions for optimal allocation choice under parameter uncertainty and uncertainty aversion by Garlappi, Uppal, and Wang (2007) – hereafter GUW – to allow for the possibility of having to liquidate assets in the future. These liquidity-adapted GUW solutions are discussed in detail in Section III below.

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4 For example, for the single-asset/single-period case, the optimal demand $X$ of an investor with a negative exponential vN/M expected utility function is $X = \frac{E(P_{t+1}) - P_t}{\gamma \text{Var}(P_{t+1})}$ (derived from the derivative of the expected utility $X(E(P_{t+1}) - P_t) - \frac{\gamma}{2} X^2 \text{Var}(P_{t+1})$ with respect to $X$, where $P_t$ is the current price of the security, $P_{t+1}$ is the final price of the security, $\gamma$ is the Arrow-Pratt coefficient of absolute risk aversion, $E(\cdot)$ denote the investors expectation, and $\text{Var}(\cdot)$ denote the variance. Thus, the demand is zero only when $E(P_{t+1}) - P_t = 0$ (i.e., when the expected return is zero). If the expected return is non-zero, a risk averse investor wishes to take a position that is increasing in the magnitude of the expected return; if the magnitude is small, the investor will still take a small position, since, under vN/M utility, the infra-marginal return exceeds the infra-marginal risk cost for small positions (i.e., $X(E(P_{t+1}) - P_t) - \frac{\gamma}{2} X^2 \text{Var}(P_{t+1}) > 0$ for $|E(P_{t+1}) - P_t| > 0$, $\text{sign}(X) = \text{sign}(E(P_{t+1}) - P_t)$, and $X$ sufficiently small). In contrast, Dow and Werlang (1992a) show that under uncertainty, there is an interval of expected return (around zero) in which an uncertainty-averse investor is “paralyzed” by the uncertainty and will not want to hold any of the security. Dow and Werlang (1992b) have shown that this paralysis can cause asset prices to have greater volatility relative to those in an economy with just risk-averse (but not uncertainty-averse) investors. (A similar result is derived in Epstein and Wang (1994), which considers a more general setting that considered in Dow and Werlang (1992).) Intuitively, given this area of paralysis, current prices must be further away from the expected future price in order to induce investors to hold the security so that the market clears.
Corresponding to the solutions in GUW, we consider four different potential optimal allocations: (1) the standard mean-variance Markowitz allocation, (2) Bayes-Stein allocation under estimation risk, (3) optimal allocation under uncertainty aversion, and (4) optimal allocation under both estimation risk and uncertainty aversion.

The liquidity-adapted GUW solutions allow us to conduct two types of analysis on the impact of real estate liquidity costs. One analysis considers how the optimal allocation to real estate depends upon the likelihood that the real estate holdings will be liquidated at the end of the period. The second type of analysis considers the ex-post performance of the four allocations schemes, simulating liquidation of real estate holdings in a manner consistent with the ex-ante probability of real estate liquidation assumed when determining the optimal allocation. It must be noted that the paper develops allocation and performance implications under an extreme assumption on liquidation that puts real estate at a disadvantage relative to other asset classes but gives us a lower bound on the contribution of real estate. In particular, we consider just the level and variability of the liquidity cost for commercial real estate while setting the liquidity costs and the probability of liquidating all other asset classes to zero. If the impact of liquidity costs is small in this extreme case, then we can safely focus on other sources of costs/risks to explain the extreme allocation/performance results. If the impact is large, then further research is required to capture cross-sectional differences in the level and variance/covariance of liquidity costs across asset classes.

In addition to the analysis described above, the paper also provides three other types of information. First, it gives a sense of the magnitude of the real estate liquidation
probability below which the diversification benefits of direct investment in real estate outweigh their higher liquidity costs. Second, it provides perspective on the magnitudes of the effects of liquidity costs, estimation risk, or uncertainty aversion and identifies the issues that are relatively more important in terms of ex-post performance. Third, by comparing the ex-post performance across allocation schemes, the paper identifies the best allocation scheme and documents the incremental improvement in ex-post performance produced by direct investment in commercial real estate under that scheme.

The results show that even when the probability of liquidation for real estate holdings is subjectively high, the impact on the allocation to commercial real estate and ex-post optimal portfolio performance associated with considering liquidation and liquidity costs is fairly small relative to that produced by uncertainty aversion. Optimally controlling for estimation risk and uncertainty aversion generates the largest drop in the optimal weight on real estate but the largest improvement in ex-post portfolio performance. Furthermore, the incremental improvement in ex-post performance associated with adding real estate to the optimal portfolios under uncertainty aversion and/or estimation risk is sizable, ranging from an increase in the ex-post Sharpe ratio of 28 percent (when the real estate liquidation probability is zero) to 9 percent (when the liquidation probability is 10 percent).

The paper is organized as follows. Section II discusses recent approaches to measuring liquidity in real estate markets that we will exploit in our study. Section III describes the liquidity-augmented asset allocations solutions based on GUW and provides the analytical solutions for optimal portfolios for the various special cases considered in
the paper. Section IV discusses the empirical estimation (including the data used) and the application of the various portfolios. This section also provides the results and compares the ex-post performance of each portfolio. Section V concludes the paper.

II. Commercial Real Estate Liquidity and the TBI

This section discusses the alternative approaches to liquidity found in the real estate literature. These approaches differ from the approaches taken in the traditional asset pricing literature mostly due to the limitations inherent in real estate data. In contrast to traditional assets classes, real estate assets are extremely heterogeneous, infrequently transacted, and not traded in markets intermediated by dealers who (almost continuously) publically quote bid and ask prices. Nonetheless, commercial real estate markets present an ideal environment in which to address the impact of illiquidity on portfolio choice beyond what is known from financial asset markets. Although there exists cross-sectional variation in liquidity across securities in traditional assets classes, the markets for these assets are generally extremely liquid (especially when compared to commercial real estate markets); the absolutely low level of liquidity costs may not produce significant effects. Data from real estate markets, however, affords us the opportunity to gauge the importance of liquidity given much more extreme cross-sectional variation in liquidity. Furthermore, commercial real estate markets provides the most detailed and well documented set of performance data of any of the real asset classes that institutional investors would typically invest in.

One stream of literature on liquidity that has developed in the commercial real estate field is that of, *inter alia*, Lin and Vandell (2007), Bond *et al* (2007), and Cheng,
Lin and Liu (2010). In contrast to the market-impact measures of liquidity used in the asset pricing literature, this real estate literature considers the volatility in asset returns over the (uncertain) time to sale (or time on market) as a measure of liquidity. When the volatility of the time to sale is taken into consideration, the *ex-ante* volatility of real estate returns can be much greater than an *ex-post* measure of volatility calculated from historical real estate returns. Thus, the use of time-on-the-market measures may help to explain why the weight institutional investors allocate to commercial real estate is lower than that typically found by naïve applications of standard portfolio optimization techniques. While this approach clearly has merit, it can be difficult to implement due to limited availability of reliable data on time-on-the-market for a sufficiently long time-series for a representative cross-section of commercial real estate properties.

An alternative liquidity measure that has been suggested for commercial real estate is a byproduct of producing a transaction-based index (TBI) of commercial real estate value based on observed market transactions of properties included in the NCREIF Property Index. As discussed next, this measure corresponds more closely to the liquidity measures used in the asset pricing literature and is, as a result, more conducive to be used in portfolio choice problems. Furthermore, one advantage of a TBI-based measure is that it is available at a quarterly frequency over an extensive period of time (from 1984 for the all property measure or from 1994 at the sector level).

5 Other reasons put forward include high management costs and transaction costs, the downward bias in volatility measures calculated from appraisal data (the appraisal smoothing problem), and insufficient investable assets being available.
For the TBI, the moments of the distributions of buyer and seller reservation prices are inferred from observed transaction prices (based on well-known techniques for inference under sample selection bias developed by Heckman (1979)). (See Fisher, Geltner and Pollakowski (2007), and Fisher et al (2003) for a detailed description of the construction of the TBI and the assumptions and methods used to infer these prices.) From this information, two indices for a representative collection of commercial real estate properties are created: the mid-point of the means of the buyer and seller reservation distributions (which is the TBI) and the mean of the buyer reservation distribution (the so-called constant-liquidity index).

Below we argue that the difference between these two indices can be interpreted as (half of) a bid-ask spread for real estate. However, a few significant points of departure from the bid-ask spreads in the market microstructure literature are important to note. First, the TBI is based on portfolios of properties, not individual assets. Second, real estate is transacted in a decentralized market, not on a centralized exchange. The TBI liquidity measure is based on estimated reservation price distributions, which are very different from liquidity measures based on differences between posted bid-ask quotes used in microstructure studies. Finally, the usual caveats about estimating transaction-based indices when few properties trade applies. However, the TBI uses the methodology of Clapp and Giacotto (1992), which may be more robust to this problem than residential price indices based solely on transactions data.

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6 We also note significant empirical challenges exist in attempting to accurately create a time-on-market measure for commercial real estate.
Nonetheless, the difference between the mean and the buyer reservation price may
be interpreted as a measure of real estate liquidity cost according to the following
argument. First consider the mean of the buyers’ reservation prices. Under the assumed
log-normal distribution for reservation prices, the mean is associated with a specific
percentile. Thus, the mean of the potential buyers’ distribution (potentially changing
over time with market conditions) is associated with a fixed probability of a seller being
able to find a suitable buyer; if the seller prices the property at this mean, then the
probability of successfully finding a buyer who is willing to pay that quoted price or
more is constant (at 50 percent). Consequently, the mean of the buyers’ distribution is
referred to as a constant liquidity index, the value that sellers can discount price to in
order to ensure a particular probability of sale (i.e., a given fixed amount of liquidity).

Next consider the mid-point. One source of differences between seller and buyer
reservation prices may be due to differences in information or beliefs (rational or
otherwise) about the value of future cash flows. In that case, the mid-point represents a
consensus fundamental value. Alternatively, some of the difference between buyer and
seller reservation prices may be due to systematic differences between the values of the
cash flows that can be generated by potential buyers (investors who, by definition, have

7 Similarly, one could develop different indices for different percentiles of the buyers’ reservation
distribution. The TBI, however, produces only the index associated with the mean (or fiftieth percentile).

8 The mid-point also has the interpretation as a variable-liquidity index. As the reservation price
distributions move (by potentially different amounts), the mid-point will correspond to different percentiles
on each reservation distribution. For example, if the buyer reservation distribution falls by an equal amount
as the seller distribution rises, even though the mid-point will stay fixed, the mid-point will be associated
with a higher percentile on the buyer reservation distribution. As a result, if a seller were to maintain an
asking price at the mid-point, the probability of finding a buyer willing to pay the mid-point or more will
have fallen with the shift in buyer reservation distribution. Thus, although the mid-point is fixed, the
liquidity (i.e., the likelihood of a transaction) at that price has fallen.
chosen not to own the property yet) versus potential sellers (investors who currently own the property). If such systematic differences exist, then the mid-point will underestimate fundamental value (with the value to next–best–use potential buyers pulling down the value); rather, the mean of the seller’s distribution would be a better measure of fundamental value (i.e., the value of the property held under its best use). Since the measure used in the paper is the difference between the mid-point and the buyer reservation mean, depending upon which interpretation is best, this measure is either correct in magnitude or half the correct magnitude. As discussed below, by considering different probabilities of liquidation, either interpretation can be made.

Under the fundamental-value interpretation of the mid-point, the difference between the mid-point and the mean of the buyer’s reservation price distribution can be viewed as a liquidity cost – the amount by which a seller would have to discount price relative to its fundamental value in order to achieve a given amount of liquidity. This is similar to half the bid-ask spread in financial security markets; the quoted bid price in financial security markets reflects the willingness of a potential buyer to pay that price (or better). However, similar to the mean of the buyer reservation price distribution, the probability of transacting a block of financial securities (similar to transacting a large real estate property) at the bid is not one. When a seller seeks to sell a block of a specific security that is larger than the quoted depth at the bid, the seller will have to sell at an average price below the bid. That is, although the probability of being able to transact a quantity equal to the quoted depth at the bid is one, the probability of being able to
transact a larger block at the bid is far less. Similarly, the probability of being able to transact at the mean of the buyers’ reservation is also less than one.

A bid-ask spread measure contrasts with the time-on-the-market liquidity measures typically used in empirical real estate analysis. However, finance theory suggests that these two measures are related (with each translating the other into a different dimension). Liquidity can be measured as either (1) the amount of time it takes to sell an item at a particular price or (2) the price concession that must be made in order to ensure the item is sold within a particular period of time. Time-on-the-market measures capture liquidity in a time domain, while bid-ask spread measures capture liquidity in a price domain. One way to view the bid-ask-spread/price-domain liquidity measure is that it is the price that makes the average investor indifferent between waiting and not waiting. An owner interested in selling has the option of either selling at the bid or waiting (as long as it takes) to sell the property at its “fundamental value.” But, since there is risk associated with waiting (i.e., the fundamental value may change), waiting is risky and costly. Alternatively, the cost of not waiting is half the spread. In order to be consistent with the set of techniques for optimal portfolio choice based on static optimization, rather than use measures of liquidity based on the time dimension, we opt to use liquidity measures in the price dimension based on TBI.

However, liquidity measures based on the TBI may misestimate real estate liquidity in the following ways. First, if the fundamental value is truly the mean of the seller reservation, then our measure is underestimated by a factor of 2. Second, the spread measure is the cost associated with liquidating a whole “representative” property;
if the fund manager holds numerous properties and must liquidate only a few properties, then the liquidity cost associated with this liquidation will be the weighted average of zero (associated with those properties not sold) and the liquidity cost of the properties sold, with weights that correspond to the value of real estate the manager continues to hold versus the value sold. Thus, the liquidity cost of partial liquidation is over-estimated by the TBI measure. As will be seen below when we examine the data, the liquidity measure derived from the TBI can be fairly large, with the highest values ranging from 25 to 35 percent. These estimates are likely to be too high for most portfolios managed by institutional investors. Thus, if we find that liquidity costs have a significant impact on both the allocation and ex-post performance, we must be careful not to over-interpret these results. However, if the allocations and ex-post performance are relatively insensitive to the inclusion of these liquidity costs (as compared, for example, to the sensitivity with respect to uncertainty aversion), then liquidity costs are likely to have relatively minor role.

III. Asset Allocation Models

As extensively justified in GUW (2007), we consider static optimization. Specifically, we consider variations of the following standard mean-variance portfolio choice problem faced by a typical mean-variance investor who (1) has a choice of $N$ securities and (2) knows the joint distribution of returns with certainty:

$$\max_{\omega} \omega' \mu - \frac{1}{2} \omega' \Sigma \omega,$$

s.t. $\omega' 1_N = 1$
where \( y \) is the Arrow/Pratt coefficient of absolute risk aversion, \( \mu \) is the (known) \( N \) by 1 vector of expected returns, \( \Sigma \) is the (known) \( N \) by \( N \) variance/covariance matrix for the \( N \) returns, \( \omega \) in the \( N \) by 1 vector of portfolio weights, and \( 1_N \) is a \( N \) by 1 column vector of 1s. With no risk-free security, the solution to (1) is

\[
\omega^{MV} = \frac{1}{\gamma} \Sigma^{-1} (\mu - \mu_0 1_N),
\]

where

\[
\mu_0 \equiv \frac{B - \gamma}{A}, \tag{3}
\]

\[
A \equiv 1'_N \Sigma^{-1} 1_N, \tag{4}
\]

\[
B \equiv \mu' \Sigma^{-1} 1_N. \tag{5}
\]

Also important in subsequent sections, the set of weights \( \omega^{MIN} \) that produce the minimum variance portfolio is given by

\[
\omega^{MIN} = \frac{1}{A} \Sigma^{-1} 1_N. \tag{6}
\]

Note that \( \omega^{MIN} \) is not a function of the mean return vector \( \mu \).

\section*{A. Return Moments and Liquidity}

In order for the above static problem to be relevant for portfolios that include investment in commercial real estate properties, returns must be defined in such a way as to capture liquidity costs and risk. Although the liquidation of commercial real estate property typically takes time, the definition of liquidity used here (which will affect estimates of both \( \mu \) and \( \Sigma \)) translates liquidity costs into a contemporaneous return/price impact dimension so that the above static model can be used. Once returns and the moments of the joint distribution of returns are redefined to include liquidity costs, the
analytical expressions for optimal weights developed in GUW (2007) as a function of the (redefined) moments can be applied.

Let the realized return on asset $i$ from time $t-1$ to $t$ (denoted $r_{it}$) be defined as

$$r_{it} = \frac{P_t^i + D_t^i - I_t^i p_{t-1}^i}{p_{t-1}^i} = \mu_i + \eta_{it} + I_t^i (\lambda_i + \xi_{it}),$$

(7)

where $P_t^i$ denotes the non-liquidation value of the asset (e.g., the mid-point of the bid-ask spread) at $t$, $D_t^i$ denotes the income flow (e.g., dividends for equity investments) between $t-1$ and $t$, $I_t^i$ denotes an indicator variable that equals 1 if the asset is liquidated at $t$ and 0 if not, $L_t^i$ denotes the dollar liquidity cost (i.e., the price concession or half the bid-ask spread) if the asset is liquidated at $t$, $\mu_i$ is the mean of $\frac{P_t^i + D_t^i - I_t^i p_{t-1}^i}{p_{t-1}^i}$ (i.e., the mean return based on non-liquidated values) while $\eta_{it}$ is the deviation of the period-$t$ realization of $\frac{P_t^i + D_t^i - I_t^i p_{t-1}^i}{p_{t-1}^i}$ from $\mu_i$, $\lambda_i$ is the mean of $\frac{L_t^i}{p_{t-1}^i}$ (i.e., the mean liquidation cost expressed in return relative to $P_{t-1}^i$) while $\xi_{it}$ is the deviation of the period-$t$ realization of $\frac{L_t^i}{p_{t-1}^i}$ from $\lambda_i$.

In matrix notation, the $N$ by 1 column vector of the $N$ asset returns is

$$r_t = \mu_r - P \lambda + \eta_t + \xi_t,$$

(8)

where $\mu_r$ is the $N$ by 1 vector of mean non-liquidation returns $\mu_i$, $\eta_t$ is the $N$ by 1 vector of $\eta_{it}$, $\lambda$ is the $N$ by 1 vector of mean liquidity costs (measured in returns), $P$ is a diagonal $N$ by $N$ matrix with the $i^{th}$ diagonal element denoting the probability that the $i^{th}$ asset will be liquidated, and $\xi_t$ is the $N$ by 1 vector with the $i^{th}$ element equal to $I_t^i \xi_{it}$.

Thus, with the possibility of liquidation, the vector of expected returns $\mu$ used in Problem (1) consists of two parts: (1) the mean return if the security is not sold at the end
of the period (i.e., $\mu_r$) and (2) the expected cost if the security is sold at the end of the period (i.e., $P\lambda$). This second expected liquidity cost component consists of two parts: the probability that the security will be liquidated $P$ and the expected cost conditional on the position being sold $\lambda$. Thus, the mean return vector $\mu$ in equation (1) is

$$\mu = \mu_r - P\lambda.$$  \hspace{1cm} (9)

The $i^{th}$ diagonal element of matrix $P$ specifies the probability that the investor will sell some fixed portion of their holdings in the $i^{th}$ asset class; the $i^{th}$ element of $\lambda$ is the impact on the realized return for the whole position in the $i^{th}$ asset class associated with the manager selling that fixed portion of the $i^{th}$ asset class.

We assume that the matrix of liquidation probabilities $P$ is fixed independent of the realized values of raw returns and liquidity costs. More realistically, the likelihood of a particular asset being liquidated will likely depend upon the outcome of some future optimization in which the investor has a need for cash and picks the set of securities to liquidate in order to meet the cash constraint at minimum cost. Then, to the extent that the need for cash and the number of securities that must be sold to satisfy the cash need will be different if the market is down than if it is up, $I_t^i$ might be correlated with the random components of realized raw returns $\eta_t$ and/or realized liquidity costs $\xi_t$. Allowing for this is beyond the scope of this paper.

The $i^{th}$ diagonal element of matrix $P$ should be thought of as the unconditional probability of having a liquidity need times the unconditional probability that, given such a need, the fund manager will liquidate an average quantity of holdings from the $i^{th}$ asset class. Thus, if the manager knows that her fund is large enough that in the event of a
liquidity need, she will never have to liquidate a position in an extremely illiquid asset (say the $k^{th}$ asset class), then the unconditional probability for that asset class should be zero (i.e., the $k^{th}$ diagonal element of the matrix $P$ is zero). If, however, the manager knows that, in the event of having a liquidity need, she will always liquidate a portion of the positions in the most liquid assets classes, then the unconditional probabilities in the diagonal element of $P$ associated with these asset classes will simply equal the probability of having a liquidity need; technically, the entry it will equal the probability of having a liquidity need times 1 (the probability of liquidation given a need). When the liquidity cost for those asset classes that the fund manager will use to generate liquidity are essentially zero, the expected liquidity cost of those assets are essentially zero. An entry in the matrix $P$ can also be non-zero if the fund manager routinely turns over assets in that class independent of having a liquidity need.\(^9\)

The cross-section of risks is a function of the randomness (i.e., the joint distribution) in the realized raw returns and liquidity costs: $\mu_t + \eta_{lt} + I^t_l(\lambda_t + \xi_{lt})$. Thus, the variance/covariance matrix $\Sigma$ used in Problem (1) is given by the following:

$$\Sigma = \Sigma_r + P'\Sigma_\lambda P - 2\Sigma_{r\lambda} P, \quad (10)$$

where $\Sigma_r$ is the $N$ by $N$ variance/covariance matrix of raw returns (excluding liquidity costs), $\Sigma_\lambda$ is the $N$ by $N$ variance/covariance matrix of liquidity costs (conditional upon

\(^9\) For example, Acharya and Pedersen (2005) consider asset pricing with liquidity costs assuming that all investors completely liquidate their holdings at the end of the next period. This corresponds to the $P$ matrix being set to the identity matrix (i.e., all of the diagonal probabilities are 1).
their being realized), and $\Sigma_{r}\lambda$ is the $N$ by $N$ matrix of the covariance between the raw return and the liquidity cost; that is

$$
\Sigma_{r}\lambda = \begin{pmatrix}
\text{Cov}(\eta_{1t}, I_{1}^{1} \xi_{1t}) & \cdots & \text{Cov}(\eta_{1t}, I_{N}^{N} \xi_{Nt}) \\
\vdots & \ddots & \vdots \\
\text{Cov}(\eta_{Nt}, I_{1}^{1} \xi_{1t}) & \cdots & \text{Cov}(\eta_{Nt}, I_{N}^{N} \xi_{Nt})
\end{pmatrix}.
$$

(11)

As in GUW (2007), we assume that the estimates of the variances/covariances are perfectly precise. However, since the estimated values of the mean raw return and mean liquidity costs will be functions of the randomness of the realized raw returns and liquidity costs, cross-sectional variation in the degree of parameter uncertainty will be a function of the variance/covariance matrix defined above.

**B. Optimal Weights: Four Cases**

For reference, the next section provides the analytical expressions derived in GUW (2007) for optimal portfolio weights for four distinct cases. Keep in mind that although GUW (2007) does not consider liquidity (and uncertainty with respect to mean liquidity costs), their expressions (reproduced below) hold in our application to commercial real estate and liquidity as long as the return moments are defined as in equations (9) through (11) above.

1. **The Benchmark Case: Classical Mean Variance Optimization (MV)**

   For the benchmark “classical” mean-variance case, it is assumed that investors behave as if their estimates of the moments of the joint distribution of returns are the
actual population moments. Given estimates for mean non-liquidation returns (denoted \( \hat{\mu} \)) and mean liquidation costs (denoted \( \hat{\lambda} \)), the mean total return vector is simply

\[
\hat{\mu} = \hat{\mu}_r - P \hat{\lambda}.
\]

Replacing \( \hat{\mu} \) (and \( \hat{\Sigma} \)) for \( \mu \) (and \( \Sigma \)) in equations (2)-(5) results in the classical mean-variance weights (based on estimated moments), which we denote by \( \hat{\omega}^{MV} \), where

\[
\hat{\omega}^{MV} = \frac{1}{y} \Sigma^{-1}(\mu - \mu_0 1_N)^{\prime}, \quad \mu_0 = \frac{B - \gamma}{A}, \quad A = 1_N^{\prime} \Sigma^{-1} 1_N, \quad B = \mu^{\prime} \Sigma^{-1} 1_N.
\]

We similarly define \( \hat{\omega}^{MIN} = \frac{1}{A} \Sigma^{-1} 1_N \) as the minimum variance portfolio weights that obtain when \( \hat{\Sigma} \) is used in equation (6). The above expressions for \( \hat{\omega}^{MV} \) and \( \hat{\omega}^{MIN} \) play a key role in the other cases discussed below since the optimal weights for all of these cases can be expressed as simple weighted averages of \( \hat{\omega}^{MV} \) and \( \hat{\omega}^{MIN} \).

2. Estimation Error (without uncertainty aversion): Bayes-Stein Estimates (BS)

If the investor understands that the estimates are not infinitely precise, then the optimal weights \( \omega^{BS} \), which derive from empirical Bayes-Stein estimates, are given by

\[
\omega^{BS} = \phi^{BS} \hat{\omega}^{MIN} + (1 - \phi^{BS}) \hat{\omega}^{MV}
\]

where

\[
\phi^{BS} = \frac{\nu \mu}{\nu + \nu \mu},
\]

\[
\nu \mu = \frac{N + 2}{(\mu - \mu_{MIN})^{\prime} \Sigma^{-1} (\mu - \mu_{MIN})},
\]

\( T \) is the number of time-series observations, \( N \) is the number of securities (as before), and \( \mu_{MIN} \) is the expected return on the minimum variance portfolio.
3. Uncertainty Aversion (without Estimation Error): Ambiguity Averse (AA)

This case corresponds to the situation in which the investor has infinitely precise estimates of the parameters characterizing the (unconditional) joint distribution, but that the returns are drawn from one of multiple known distributions, where the specific distribution realized is drawn from a known (and non-trivial) distribution of distributions. In this case, there is no risk or uncertainty created by estimation error. If the investor is just risk averse (and not also uncertainty averse), this case corresponds to MV above. However, if, in addition to being risk averse (i.e., \( \gamma > 0 \)), the investor is also uncertainty averse (or is ambiguity averse), the investor’s optimal weights must take into consideration how the ambiguity in the distribution affects the desirability of specific assets. In this case, the optimal weights \( (\omega^{AA}) \) are given by

\[
\omega^{AA} = \phi^{AA} \omega^{MIN} + (1 - \phi^{AA}) \omega^{MV}
\]  

(16)

where

\[
\phi^{AA}(\varepsilon) = \frac{\sqrt{\frac{(T-1)N}{T(T-N)}}}{\gamma \sigma_p^* \sqrt{\frac{(T-1)N}{T(T-N)}}}
\]

(17)

and \( \sigma_p^* \) is the value of \( \sigma_p \) that solves the following polynomial:

\[
A \gamma^2 \sigma^*_p + 2A \gamma \sqrt{\varepsilon} \sigma^3_p + (\varepsilon - AC + B^2 - \gamma^2) \sigma^2_p - 2\gamma \sqrt{\varepsilon} \sigma_p - \varepsilon = 0,
\]

with \( \varepsilon = \frac{\varepsilon (T-1)N}{T(T-N)} \) and \( C = \hat{\mu}' \hat{\Sigma}^{-1} \hat{\mu} \). The value of \( \varepsilon \) (and thus \( \varepsilon \)) is chosen by the investor given the level of confidence he/she wants to impose on the estimates of the expected returns. Specifically, \( \varepsilon \) is value on an F distribution with \( N \) and \( T-N \) degrees of freedom associated with the level of confidence sought by the investor.
4. **Estimation Error and Uncertainty Aversion (BSAA)**

Finally, if the moments are estimated with error and the investor has ambiguity aversion, then the optimal weights ($\omega_{AA}^{BS}$) are

$$\omega_{AA}^{BS} = \phi^{AA}(\varepsilon)\omega^{MIN} + (1 - \phi^{AA}(\varepsilon))\omega^{BS}. \quad (18)$$

**IV. Empirical Estimation**

This section discusses the results of two types of empirical analysis. First, optimal portfolio weights for the four cases discussed in Section III are calculated. The second type of analysis then examines the ex-post performance of these portfolios. By examining the differences in allocation weights across asset classes and the ex-post performance for each of the four portfolios outlined in Section III, we identify which issues (estimation risk, liquidity, or uncertainty aversion) are relatively more important.

In addition to commercial real estate, we also consider three other asset classes: Large capitalization equities are represented by the S&P500 index, small capitalization companies are tracked using the Russell 2000 index, and Government bonds are represented by the government bond index obtained from JP Morgan. As the TBI real estate data is only available quarterly, all series have been converted to quarterly returns using index values at the end of each quarter. The longest period in which all data was available is from 1988, quarter 2, to 2010, quarter 1. Table 1 presents summary statistics.

[Insert Table 1 here]

Over the sample period, small capitalization stocks had the highest quarterly average return, followed by real estate and bonds. Large capitalization stocks had the
lowest annual return over this period. Small capitalization stocks also had the highest volatility followed by real estate, large capitalization equities, and bonds. Real estate returns display considerable excess kurtosis.

As explained in Section II, the liquidity cost measure is the difference between the TBI and the mean buyer reservation price obtained from NCREIF transactions-based real estate index obtained from the MIT Center for Real Estate. To allow for the TBI liquidity measure to represent a cost of liquidation, we rebase the index so that the liquidity cost takes the value of zero during the period of highest market liquidity, in this case the third quarter 2005 (see Figure 1). The justification for this choice is that during this time it was very common for commercial buildings to trade with no appreciable reduction in value for a quick sale. In most other time periods, liquidation of an asset over a short time period (even allowing for a quarterly time period) would typically result in a lower offer than that which would be possible over a longer marketing period.10

[Insert Figure 1 here]

The average liquidity cost over the sample period was 15.9%, with a standard deviation of 1.2%. That is, the cost of immediate liquidation was 15.9% on average. However, there is a great deal of variability in this liquidity measure; at times there was essentially no liquidity cost (e.g., the mid-2000s), whereas in other periods, such as in the recent “great” recession, the costs of immediate liquidation could be very large (which is consistent with anecdotal evidence from foreclosure sales).

Here we do not consider liquidity costs for the other asset classes because, relative to real estate, equities and government bonds are very liquid, being traded in highly liquid centralized exchanges. While liquidity costs can be important in both of these markets, we believe that the quarterly time period used in this study helps to alleviate this problem. However, a future extension of this research will consider liquidity measures such as those suggested by Amihud (2002), which could be calculated for each asset class.

As inputs into the allocation models, we estimate the expected returns and the variance-covariance matrices using 40 quarter (10 years) rolling windows. The risk aversion parameter $\gamma$ is set to one. Also, consistent with GUW, we do not include a risk-free asset class. This is consistent with the notion that, for a multi-asset portfolio, cash investments tend to be held for liquidity purposes rather than as a specific asset choice. For comparison we did estimate optimal portfolio holdings including a treasury bill, however, we found that due to the low variation in the interest rate, very high allocations were associated with the risk-free asset (results not reported).

### A. Results – without liquidity costs

We first consider the asset allocations without including liquidity costs or constraining the weights to fall between zero and one. Figure 2 shows the optimal portfolio weights for each asset class across time. As is typically seen in such analysis, a very high weight is given to real estate. Indeed the mean-variance and Bayes-Stein estimates are quite extreme, suggesting leveraged positions in real estate and small capitalization stocks, and a consistent short position in large capitalization stocks. Only the cases with uncertainty aversion (AA and BSAA) look reasonable.
The ex-post Sharpe ratios for each allocation model are shown in Table 2. For any quarter $t$, the inputs needed for the optimal portfolio are estimated using the data on returns from ten years prior to $t$ to $t$. Then, using these inputs, the optimal portfolio to be held during quarter $t$ is constructed. Given this portfolio and the returns that were realized from the beginning of quarter $t$ (when the portfolio was formed) to the end of quarter $t$, the return on that portfolio for quarter $t$ is calculated. This is done for all quarters starting ten years after the beginning of the data. For each quarter, the risk-free rate during that quarter is subtracted from the realized portfolio return for that quarter. With this series of excess returns, the average and the variance of the realized excess returns are calculated. Using these averages and variances, ex-post Sharpe ratios for each type of portfolio are calculated. Table 2 shows that the portfolios that incorporate both estimation risk and uncertainty aversion have the highest Sharpe ratios. The Shape ratios associated with portfolios with extreme allocations to real estate (e.g., MV and BS) indicate extremely poor performance, even without considering liquidity costs.

The extreme portfolio allocations suggested by the unconstrained allocation methods, while possible when implemented using derivative strategies, are unlikely to be followed in practice. Hence we repeat this exercise allowing for no short sales or leverage positions in the respective asset classes. Figure 3 shows the results of these constrained allocation models.
Overall the constrained estimates look much more reasonable and, in some instances, are not dissimilar to the type of portfolios that may be held by institutional investors. However, the mean-variance weights still show extreme positions (indicated by the labels MV in Figure 3). There are no holdings in large capitalization stocks and almost no allocation to bonds. The mean-variance portfolio is almost entirely invested in small capitalization stocks until 2002 and then the allocation shifts to 100% in real estate until the end of 2009.

The Bayes-Stein model shows a similarly high allocation to real estate; although this is not as extreme as the mean-variance results, it is still much larger than many institutional investors hold in their portfolios (weights are shown with the label BS). The holdings in real estate are supplemented by an allocation to bonds and a small allocation to large capitalization equities.

Finally the two models incorporating uncertainty aversion produce generally similar allocations. In each, the allocation to real estate is similar and much lower than generally found (around 10-12% by the end of the sample period). It is noticeable that in the uncertainty aversion model the allocation to bonds is much higher than typically found in other multi-asset allocation studies.

Table 3 shows the ex-post Sharpe Ratios and the uncertainty aversion models dominate the mean-variance and Bayes-Stein models. This table also provides the ex-post Sharpe ratios for each of the four cases when commercial real estate is not included in the portfolio. In each case, the inclusion of commercial real estate assets improves portfolio performance. But note that even though, out of the four cases, the weights
placed on commercial real estate are the smallest under uncertainty aversion cases (see AA and BSAA in Figure 3), the incremental improvement to ex-post performance associated with including real estate is the largest under uncertainty aversion (with the ex-post Sharpe ratios rising by more than .1).

[Insert Table 3 here]

This table also shows the difference in the ex-post performance of the optimal portfolios containing direct investment in commercial real estate versus the optimal portfolios that invest in commercial real estate via REITs. It is interesting to note that the joint distribution of returns when adding only direct investment in the commercial real estate represented in the TBI is such that there is a significant ex-post performance improvement (from a Sharpe ratio for the BSAA portfolio of .401 excluding real estate of any kind to a Sharpe ratio for the BSAA portfolio of .516 when adding direct investment in commercial real estate – 28 percent higher), while no such improvement exists when adding real estate via REITs.\textsuperscript{11} Since this table does not include liquidity costs (which will be higher for direct investments as compared to liquefied investment via REITs), this result suggests either that (1) the implicit security weights implied by the REIT index are more sub-optimal than those implied by the TBI index (and, as a result, produce relatively worse ex-post performance), or (2) that there is a systematic difference in the return characteristics (e.g., correlation with other securities and degree of estimation risk and parameter uncertainty) between those real estate assets that are held in REITs versus

\textsuperscript{11} Since these are ex-post performance measures, it need not be the case (as with Sharpe ratios based on expectations) that the Sharpe ratios increase when a larger set of assets is considered.
those that are not liquefied and must be held directly (i.e., those captured in the TBI), with direct investment in commercial real estate providing superior diversification benefits. Below, when liquidity costs are considered, a comparison of average Sharpe ratios including direct – but illiquid – ownership of real estate and ownership via liquefied REITs can be made.

**B. Results – with liquidity costs**

Using the methods outlined in Section III, we incorporate a measure of liquidity cost and the likelihood of incurring that cost into the allocation decision. Focusing specifically on the allocation to real estate, Figure 4 shows the portfolio weights estimated from each portfolio construction method as the probability of liquidation takes values from $P=0$ (no liquidation considered, hence this coincides with the estimates in the previous section) to $P=0.50$ (there is a 50 percent likelihood the real estate portfolio is liquidated at the end of each quarter – an admittedly extreme position). Note that since the liquidity costs are set to zero for all of the asset classes other than commercial real estate, the only probability in the $P$ matrix that matters is that associated with commercial real estate. Thus, we will simply refer to the liquidation probability $P$; it should be understood that this is the ex-ante probability associated with liquidating positions in commercial real estate (after positions in other assets classes have also been liquidated).

A reduction in the portfolio allocation to real estate is evident in this graph as the probability of liquidation increases. Of notable interest are the large swings in the mean-variance allocations (which fall to zero as $P$ increases), relative to the alternative methods. Combining uncertainty aversion, with estimation risk and liquidity costs, leads
to estimates of portfolio allocations that are closer to those observed in practice than many previous studies have shown. The portfolio allocations also tend to be more stable when uncertainty aversion is considered.

[Insert Figure 4 here]

The ex-post Sharpe ratios for the portfolios construction methods are shown in Table 4. The table groups the Sharpe ratios into columns according to the assumed ex-ante probability of liquidation. Note that it is not the case that the liquidation costs are actually incurred in these calculations. We address this issue below. In all cases, the uncertainty aversion models have higher ex-post portfolio performance. This difference is particularly noticeable for liquidation probabilities in the range of $P=0.01$ to $P=0.10$.

[Insert Table 4 here]

**C. Simulation of liquidity costs in Sharpe Ratio**

In the previous section, the ex-post Sharpe ratios were calculated based on realized raw returns, but not realized liquidity costs. However, if a fund manager does need to liquidate some holdings, the portfolio’s ex-post performance will not reflect any value lost due to the incursion of liquidity costs. To accommodate the impact of realized liquidity costs on portfolio performance, in this section we reproduce ex-post Sharpe ratios for portfolios where investors are randomly required to liquidate their real estate holdings (and hence incur the costs of liquidation). The number of times they may be required to do this is consistent with the probability of liquidation assumed when the portfolios are constructed (e.g., $P = 0.05, 0.10$).
The sample period used to estimate the portfolio inputs is the same as for the other portfolio calculations shown above. That is, the mean return vector and the variance/covariance matrix are estimated using the data on returns from a ten-year rolling window prior to each quarterly period.

However, for each level of ex-ante liquidation probability we simulate the liquidation event randomly over the sample period and this is repeated for 1,000 replications for each level of $P$. For example, if $P = 0.01$ and given that there are 48 quarters over which we construct the portfolio allocation, we randomly select four or five quarterly observations out of the 48 and in these quarters we set the return for real estate to be the equal to the quarterly return for real estate less the cost of liquidation. We then average the ex-post Sharpe ratios across all the replications. In this paper we only focus on the more likely liquidation probabilities of $P = 0.01$ and $P = 0.05$. The simulated ex-post Sharpe ratios are shown below in Table 5.

[Insert Table 5 here]

These results show that when liquidity costs are incurred ex-post, the uncertainty-averse portfolios continue to have the highest ex-post Sharpe ratios. By comparing the results in column 1 of Table 3 (corresponding to portfolios that exclude commercial real estate of any kind) to the results provided in Table 5, the incremental impact of including commercial real estate when estimation error, liquidity costs, and uncertainty aversion are

\[\text{In the case of } P=0.1, \text{ we select four quarterly observations out of the sample of 48 for 200 repetitions and then we select five quarterly observations for 800 repetitions. This provides an average of 4.8 across 1,000 repetitions. We only estimate the Sharpe Ratio over 1,000 repetitions, which is a small number given the possible combinations of selecting five random quarters out of 48 (which is } \binom{48}{5} = 1,712,304 \text{ possible selections).}\]
considered can be seen. While the ex-post performance for each of the four cases is poorer when the probability of forced liquidation is 10 percent for real estate (what we believe is an extremely high probability), the ex-post Sharpe ratios for the uncertainty aversion cases when the probability of liquidation is .05 (also very high) are on par with the ex-post Sharpe ratios when commercial real estate is excluded. Thus, even though commercial real estate properties have high liquidity costs (conditional upon liquidation), if the unconditional probability of needing to liquidate such properties over the next period is less than 5 percent, then adding commercial real estate improves portfolio performance – even in the face of estimation error and uncertainty aversion. If there is no uncertainty aversion (as in the MV and BS cases), then additions of commercial real estate improve performance – but maintains absolutely poor performance. Also, recall that the BSAA REIT only case in Table 3 produces an ex-post Sharpe ratio of .37. This figure is slightly less than the ex-post Sharpe ratio using only direct investment in high-liquidity cost commercial real estate when the ex-ante and ex-post probability of liquidation is 5 percent (which produces an ex-post Sharpe ratio of .396 in Table 5). Thus, if the probability of liquidation is less than 5 percent, then direct investment in commercial real estate is justified on the basis of ex-post portfolio performance.

V. Conclusions

In this paper we outline an approach for incorporating liquidity costs into portfolio allocation decisions. Importantly, the approach we employ (relying on results developed in Garlappi, Uppal and Wang (2007)) also allows for uncertainty regarding the inputs into the allocation process. In the current and recently past current economic
environment, with significant uncertainty about market liquidity and mean returns, we believe that such an approach is particularly valuable.

The first set of results in the paper demonstrates that portfolios that optimally respond to uncertainty aversion produce economically significant improvements in ex-post Sharpe ratios. In addition, the optimal consideration of uncertainty aversion also results in allocations to real estate that are much lower than have been shown in many studies in the past. Further, the extreme portfolio positions associated with mean-variance portfolio construction are naturally avoided.

The results also show the impact of estimates of liquidity cost obtained from the work of Fisher et al (2007). While this measure of liquidity is not commonly used in studies of market liquidity, we document a number of advantages in using the measure. In particular, we find that the resulting portfolio estimates are plausible and consistent with the observed asset allocation holdings of institutional investors. However, when investors do actually incur liquidation costs, the uncertainty aversion portfolios are shown to have economically significantly better ex-post performance. Thus, although the probability of incurring the high liquidity costs associated with commercial real estate asset was considered to be very high, the consideration of liquidity costs has a much smaller impact on the optimal weights and ex-post portfolio performance than does the optimal consideration of uncertainty aversion.
References


Table 1
Summary Statistics
Quarter 2 1988 - Quarter 1 2010

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Median</th>
<th>Std Dev</th>
<th>Min</th>
<th>Max</th>
<th>Skewness</th>
<th>Kurtosis</th>
</tr>
</thead>
<tbody>
<tr>
<td>S&amp;P 500</td>
<td>0.0060</td>
<td>0.0097</td>
<td>0.0404</td>
<td>-0.1100</td>
<td>0.116</td>
<td>-0.3815</td>
<td>3.6464</td>
</tr>
<tr>
<td>Russell 2000</td>
<td>0.0232</td>
<td>0.0342</td>
<td>0.1029</td>
<td>-0.2651</td>
<td>0.2936</td>
<td>-0.3359</td>
<td>3.7920</td>
</tr>
<tr>
<td>Bond</td>
<td>0.0178</td>
<td>0.0159</td>
<td>0.0220</td>
<td>-0.0270</td>
<td>0.0787</td>
<td>0.2728</td>
<td>2.5095</td>
</tr>
<tr>
<td>TBI</td>
<td>0.0184</td>
<td>0.0197</td>
<td>0.0452</td>
<td>-0.1669</td>
<td>0.1905</td>
<td>-0.4167</td>
<td>7.1169</td>
</tr>
</tbody>
</table>

Table shows the summary statistics for the four asset-class returns used in the study. The data are quarterly total returns: S&P 500 returns are the Standard and Poors 500 index to represent large capitalization equities; Russell 2000 represents small capitalization stock returns; Bond represents total returns on the JP Morgan Government Bond index; and, TBI represents total returns on the Transactions-Based Index for Commercial Real Estate Returns compiled by the MIT Center of Real Estate. All data was obtained from Bloomberg with the exception of the TBI Index which was downloaded from the website http://web.mit.edu/cre/research/credl/tbi.html.
Table 2

Ex-post Sharpe Ratios
(without liquidity costs and constraints)

<table>
<thead>
<tr>
<th>Method</th>
<th>Sharpe Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean-Variance</td>
<td>0.127</td>
</tr>
<tr>
<td>Bayes-Stein</td>
<td>0.117</td>
</tr>
<tr>
<td>Uncertainty Aversion</td>
<td>0.211</td>
</tr>
<tr>
<td>Uncertainty Aversion with Estimation Error</td>
<td>0.271</td>
</tr>
</tbody>
</table>

The table above shows the ex post Sharpe Ratios for each set of portfolios outcomes calculated using the four methods outlined in Section III. In each case the mean return vector and the variance/covariance matrix are estimated using the data on returns from a ten-year rolling window prior to each quarterly period. Given these inputs, the optimal portfolio to be held during quarter $t$ is constructed. The total portfolio return is calculated based on the optimal portfolio weights from the beginning of quarter $t$ (when the portfolio was formed) to the end of quarter $t$, the return on that portfolio for quarter $t$ is calculated. This is done for all quarters starting ten years after the beginning of the data. For each quarter, the risk-free rate during that quarter is subtracted from the realized portfolio return for that quarter. With this series of excess returns, the average and the variance of the realized excess returns are calculated. Then, using these averages and variance, the ex-post Sharpe ratios are calculated for each type of portfolio.
Table 3

Ex-Post Sharpe Ratios Excluding and Including Real Estate as an Asset Class
(short sales and leverage positions not allowed)

<table>
<thead>
<tr>
<th></th>
<th>Excluding Real Estate</th>
<th>Including Only Direct Real Estate</th>
<th>Including only REITs</th>
<th>Including both Direct Real Estate and REITs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean-variance</td>
<td>-0.088</td>
<td>-0.022</td>
<td>-0.090</td>
<td>-0.020</td>
</tr>
<tr>
<td>Bayes Stein</td>
<td>-0.034</td>
<td>0.091</td>
<td>-0.027</td>
<td>0.112</td>
</tr>
<tr>
<td>Uncertainty aversion</td>
<td>0.376</td>
<td>0.488</td>
<td>0.356</td>
<td>0.477</td>
</tr>
<tr>
<td>Uncertainty aversion with estimation error</td>
<td>0.401</td>
<td>0.516</td>
<td>0.370</td>
<td>0.496</td>
</tr>
</tbody>
</table>

The columns in the table above show the ex-post Sharpe Ratios for each set of portfolios outcomes calculated using the four methods outlined in section III. In the first column real estate is excluded in the asset set under consideration. In the second column direct real estate (as represented by the TBI index) is included as an asset class. The third column substitutes indirect real estate investment using REITs in place of holding direct property portfolios, and the fourth column includes both direct and indirect real estate exposure. In each case the mean return vector and the variance/covariance matrix are estimated using the data on returns from a ten-year rolling window prior to each quarterly period. Given these inputs, the optimal portfolio to be held during quarter $t$ is constructed. The total portfolio return is calculated based on the optimal portfolio weights from the beginning of quarter $t$ (when the portfolio was formed) to the end of quarter $t$, the return on that portfolio for quarter $t$ is calculated. This is done for all quarters starting ten years after the beginning of the data. For each quarter, the risk-free rate during that quarter is subtracted from the realized portfolio return for that quarter. With this series of excess returns, the average and the variance of the realized excess returns are calculated. Then, using these averages and variance, the ex-post Sharpe ratios are calculated for each type of portfolio.
Table 4

Ex-post Sharpe Ratios across Portfolio Construction Methods
Allowing for changes in the Probability of Liquidation

<table>
<thead>
<tr>
<th></th>
<th>P=0.0</th>
<th>P=0.01</th>
<th>P=0.05</th>
<th>P=0.10</th>
<th>P=0.50</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean-variance</td>
<td>-0.022</td>
<td>-0.014</td>
<td>0.017</td>
<td>0.003</td>
<td>0.005</td>
</tr>
<tr>
<td>Bayes Stein</td>
<td>0.091</td>
<td>0.102</td>
<td>0.121</td>
<td>0.064</td>
<td>0.016</td>
</tr>
<tr>
<td>Uncertainty aversion</td>
<td>0.488</td>
<td>0.494</td>
<td>0.487</td>
<td>0.400</td>
<td>0.064</td>
</tr>
<tr>
<td>Uncertainty aversion with estimation error</td>
<td>0.516</td>
<td>0.518</td>
<td>0.506</td>
<td>0.438</td>
<td>0.070</td>
</tr>
</tbody>
</table>

The table above shows the ex-post Sharpe Ratios for each set of portfolios outcomes calculated using the four methods outlined in Section III. The difference between this table and Table 3 is that the probability of liquidation of the real estate holdings is permitted to vary from P=0.0 to P=0.5. It should be noted that when P=0.0, the ex-post Sharpe Ratios are identical to those reported in Table 3 (when liquidity costs were not taken into consideration). Further discussion of P can be found in Section III (a). Note that in each case the mean return vector and the variance/covariance matrix are estimated using the data on returns from a ten-year rolling window prior to each quarterly period. Given these inputs, the optimal portfolio to be held during quarter \( t \) is constructed. The total portfolio return is calculated based on the optimal portfolio weights from the beginning of quarter \( t \) (when the portfolio was formed) to the end of quarter \( t \), the return on that portfolio for quarter \( t \) is calculated. This is done for all quarters starting ten years after the beginning of the data. For each quarter, the risk-free rate during that quarter is subtracted from the realized portfolio return for that quarter. With this series of excess returns, the average and the variance of the realized excess returns are calculated. Then, using these averages and variance, the ex-post Sharpe ratios are calculated for each type of portfolio.
The above table takes into consideration that investors create optimal portfolios based on the assumptions that they may need to incur liquidation costs associated with their real estate holdings. Unlike Table 4, where the Sharpe Ratios are reported without the liquidation costs being incurred, this table reports the ex-post Sharpe Ratios when investors experience liquidation events at the rate equal to the ex-ante probability of liquidation (P). In the table above, we calculate these Sharpe Ratios for P = 0.05 and 0.10. That is, of the 48 portfolios created from the data sample, in five and ten percent of these portfolios, the investors receive the quarterly real estate return less the liquidation cost. We repeat this exercise 1,000 times to consider different random draws for when the liquidation costs is incurred. The ex post Sharpe Ratios reported above are average over the 1,000 repetitions.

<table>
<thead>
<tr>
<th></th>
<th>P=0.05</th>
<th>P=0.10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean-variance</td>
<td>-0.007</td>
<td>-0.020</td>
</tr>
<tr>
<td>Bayes-Stein</td>
<td>0.080</td>
<td>0.023</td>
</tr>
<tr>
<td>Uncertainty aversion</td>
<td>0.380</td>
<td>0.232</td>
</tr>
<tr>
<td>Uncertainty aversion with estimation error</td>
<td>0.396</td>
<td>0.255</td>
</tr>
</tbody>
</table>
The figure above displays a time series plot of the real estate liquidity cost measure used in this study to represent the cost of liquidation of real estate assets within each quarter. The data is obtained from the MIT Center for Real Estate and is rebased to September Quarter 2005. The index is rebased to September Quarter 2005 as this was a time of very high liquidity in the commercial real estate market (it coincides with a peak in transactions activity). The measure is calculated as the difference between the TBI Demand Index (constant liquidity) and the TBI Supply Index, divided by the TBI Price level index at each quarter during the sample.
Figure 2

Portfolio Allocation Weights
(without constraints or liquidity costs)

For each asset class, the figure above shows the optimal portfolio allocations using the methods outlined in Section III. In this exercise no constraints are placed on the value that the portfolio weights can take (that is there are no short sale or leverage constraints applied). Similarly, we do not consider liquidity costs in this exercise. Means, variances and covariances are calculated from a 40 quarter rolling average window prior to quarter $t$. The lines labeled MV show the portfolio allocations calculated using mean-variance optimization, BS represents the portfolio allocations calculated using the Bayes-Stein method, AA represents the uncertainty (ambiguity) aversion portfolio weights, and BSAA represents the uncertainty (ambiguity) aversion estimates adjusted for estimation error. The panel labeled SP500 is the allocation under each of the four allocation methods for the portfolio holding of large capitalization equities (represented by the S&P 500 index). The panel labeled Russell 2000 shows the allocation to small capitalization stocks, the returns being represented by the Russell 2000 index. The panel labeled Bond, shows portfolio allocations calculated under the four optimization methods for the portfolio holdings in Government Bonds (represented by returns on the JP Morgan Government Bond Index). Finally, the panel labeled TBI shows the optimum portfolio holdings for commercial real estate calculated under each method listed above.
Figure 3

Portfolio Allocations
No short sales or leveraged positions allowed
(without liquidity costs)

For each asset class, the figure above shows the optimal portfolio allocations using the methods outlined in Section III. In this exercise constraints are placed on the value that the portfolio weights can take (that is the portfolio manager cannot short or take leveraged positions in any asset class). Liquidity costs are not considered in this exercise. Means, variances and covariances are calculated from a 40 quarter rolling average window prior to quarter t. The lines labeled MV show the portfolio allocations calculated using the mean-variance optimization method, BS represents the Bayes Stein portfolio weights, AA represents the uncertainty (ambiguity) aversion portfolio weights, and BSAA represents the uncertainty (ambiguity) aversion estimates adjusted for estimation error. The panel labeled SP500 is the allocation under each of the four allocation methods for the portfolio holding of large capitalization equities (represented by the S&P 500 index). The panel labeled Russell 2000 shows the allocation to small capitalization stocks, the returns being represented by the Russell 2000 index. The panel labeled Bond, shows portfolio allocations calculated under the four optimization methods for the portfolio holdings in Government Bonds (represented by returns on the JP Morgan Government Bond Index). Finally, the panel labeled TBI shows the optimum portfolio holdings for commercial real estate calculated under each method listed above.
The figure above shows the optimal portfolio allocations to the real estate asset class using the four methods outlined in Section III. The probability of liquidation is allowed to vary from $P=0.0$ to $P=0.5$. That is, the ex-ante probability that the real estate holdings need to be liquidated in a given quarter ranges from no expectation of liquidation to a 50% likelihood of liquidation. Further discussion of $P$ can be found in Section III (a). In this exercise constraints are placed on the value that the portfolio weights can take (that is the portfolio manager cannot short or take leveraged positions in any asset class). Means, variances and covariances are calculated from a 40 quarter rolling average window prior to quarter $t$. The upper-left panel shows the portfolio weights for real estate calculated using mean-variance optimization. The lower-left panel represents the Bayes-Stein portfolio weights for real estate calculated using mean-variance optimization. The upper-right side panel represents the uncertainty (ambiguity) aversion portfolio weights for real estate and the lower-right panel represents the uncertainty (ambiguity) aversion estimates adjusted for estimation error.