ABSTRACT. An issuer seeks to liquidate all or part of a portfolio of heterogeneous assets. The payoff of each asset is a monotone function of a common underlying cash flow. The issuer sees a private signal of this cash flow and decides which proportions of her assets to sell. In contrast with the prior literature, a higher signal can lead the issuer to sell more of a given asset at a higher price, as long as her total asset sales revenue falls. If the assets can be ordered in terms of informational sensitivity, the issuer sells its least informationally sensitive assets first. Under an additional hazard rate ordering assumption, these are its more senior assets, using a notion of seniority that is weaker than prior notions. Moreover, if the issuer can design any set of monotone ex ante and ex post securities, the optimal strategy (assuming the hazard rate ordering property) is to design a single ex post simple debt security whose face value is decreasing in the signal. The face value of this debt is lower, and thus the retained portion is larger, when the cash flow displays more asymmetric information, as has been observed in the case of no-documentation loans.
Introduction

This paper studies how a privately informed issuer should sell claims to a cash flow. For instance, a nonfinancial corporation can issue debt, equity, and hybrid securities, secured by its future operating profits about which it naturally has superior information. A bank can tranche and sell a portfolio of loans to borrowers about whose repayment probabilities the bank has private knowledge.

We assume the issuer needs cash more than investors. For instance, a firm may need cash to invest in a worthwhile project; a bank may need funds in order to issue more loans. Hence, under symmetric information the issuer would sell her entire future cash flow to investors. In contrast, under asymmetric information the issuer faces a lemons problem: if she sells her entire cash flow, investors may conclude that she expects it to be low. By retaining a portion of the flow, she can credibly signal that she has received positive information and thus obtain a higher price for the portion that she sells.

Most of the prior literature on this problem has assumed the issuer sells a single asset to investors. Examples include Leland and Pyle (1979), Myers and Majluf (1984), and DeMarzo and Duffie (1999). In contrast, DeMarzo (2005) considers multiple assets with multidimensional information. He shows that under certain assumptions, the quantity sold of each asset can be used to signal a different dimension of the issuer’s information, whence the single-asset solution holds separately for each asset.

In this paper we relax DeMarzo’s (2005) assumption that there are fewer assets than dimensions of private information. The resulting problem is complex as the issuer has multiple assets with which she can signal a given change in her information. To simplify, we focus on the case in which the issuer’s information is one-dimensional. This assumption is likely to be a good first-order approximation to real-world situations, and is common in applied models (e.g., Frankel and Jin 2015 [ANY OTHERS?]).

We also make a monotonicity assumption: an increase in the issuer’s information is associated with a higher payout of each asset. This holds, for instance, if higher information is associated with a higher cash flow and the issuer’s securities are
monotone. Such securities are common and include equity and standard debt, as well as the senior, mezzanine, and equity tranches of most loan pools.

In section 1, we assume the issuer has a fixed set of assets to sell. We show that there is a unique equilibrium outcome that satisfies even the mildest restriction on beliefs: the Intuitive Criterion (Cho and Kreps 1987). This criterion is easy to motivate: beliefs after a deviation must be concentrated only on types that could possibly hope to gain. However, it does not always ensure uniqueness. Its strength in our setting hinges on the ability of the issuer to underprice assets and to ration their allocation, even though in equilibrium no underpricing occurs. We show, moreover, that the unique outcome can be computed recursively.

The existing literature finds that when an issuer becomes more optimistic about an asset’s payout, she sells less of the asset at a higher price. In section 1.3 we show that this may not hold in our setting: a more optimistic issuer may sell more of an asset at a higher price. This occurs when her increased optimism can be signaled most efficiently by selling more of the given asset while selling less of some other asset whose expected value is more sensitive to her change in information.1

In section 2 we show that, in our setting, an issuer always gains from splitting a given asset into separate securities that have different informational sensitivities. Intuitively, by separately adjusting the proportions of each resulting security, the issuer can signal its information more efficiently. This implies that an optimal split will involve a portfolio of securities that is “extreme” in the space of feasible designs, in the sense that it cannot be replicated by another portfolio; one example is debt and equity.2

Section 3 returns to the case of a fixed set of assets analyzed in section 1. We define one asset as more informationally sensitive than another if an increase in the issuer’s type raises the expected payout of the first asset proportionally more than that of the second. The assets in an issuer’s portfolio are informationally ordered if they can be ranked from

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1 Despite this possibility, higher types always earn lower securitization revenue as in the single-asset case (e.g., DeMarzo and Duffie 1999). Why? If a high type had higher asset sales revenue than a low type, the low type would imitate her, thus receiving the same increase in sales revenue as the high type while facing a lower opportunity cost of parting with her assets.

2 This assumes monotone securities and limited liability. In low states, equity pays zero which cannot be lowered by limited liability; in high states, the payoff to debt equals the face value and cannot be lowered by monotonicity.
least to most informationally sensitive. A strong prediction is obtained in this case: an issuer will sell an asset only after all of its less informationally sensitive assets have been sold.

As an application, in section 4 we define a new notion of seniority that applies to both common corporate securities as well as to the more complex tranches of asset backed securities. The notion is simple: security A is senior to security B if the ratio of security A’s payout to that of security B is nonincreasing in the underlying cash flow. We show that if the securities can be ranked in this way, and if the issuer’s information satisfies the Hazard Rate Ordering property (which is weaker than the commonly assumed monotone likelihood ratio property), then the issuer’s securities can be informationally ordered. In this case, by results in section 3 the issuer adopts a seniority threshold: she sells only those assets whose seniority lies above the threshold, as predicted by Myers’ (1984) Pecking Order Hypothesis. Moreover, the seniority threshold is an increasing function of the issuer’s type. Hence, when an investor sees an issuer raise her seniority threshold (which involves retaining more assets), he knows the issuer’s type is higher. He thus raises the price he is willing to pay for the assets that the issuer still sells.

In section 5, we turn to the security design problem. In prior work, DeMarzo and Duffie (1999) studied the ex-ante case, single security case: the issuer designs a single monotone security, learns her type, and decides what proportion of her security to sell. We study the general case. The issuer first designs any number of monotone ex-ante securities. She then learns her type and designs any number of ex-post securities, which are monotone functions of the payouts of the ex-ante securities. This flexible framework includes DeMarzo and Duffie (1999) as a special case, as well as many other types of securitization schemes.

We show, first, that an optimal strategy for the issuer is to design a single security after she sees her information. Furthermore, if her information satisfies the Hazard Rate Ordering property, her optimal security is standard debt with a face value that is

3 Empirical support for the pecking-order hypothesis appears in Fama and French (2002), Flannery and Rangan (2006), Opler et al (1999), and Shyam-Sunder and Myers (1999). These papers also find some support for the “static tradeoff” model, in which a firm has an optimal debt to equity ratio that is determined partly by tax policy. Thus, informational asymmetries seem to be one of several factors that affect a firm’s capital structure.
decreasing in her information. Moreover, this strategy is equivalent to the issuer’s maximally tranching her cash flow \textit{ex ante} and then, after learning her type, selling those tranches whose seniority exceeds a threshold that is increasing in her type.

While simple debt is also optimal in the \textit{ex ante} security design case (DeMarzo and Duffie 1999), there is a key difference. In the \textit{ex ante} case, an issuer signals higher information by retaining more of a fixed debt security that it designed previously. In the general case, she instead designs a debt security with a lower face value. Since fewer constraints hold in the general case and the Intuitive Criterion selects the most efficient outcome, this must be a more efficient way to signal information. Intuitively, selling fewer shares of a fixed debt security lowers the payout to investors by the same proportion for any cash flow. In contrast, lowering the face value reduces the payout only when the underlying cash flow exceeds the face value, which is more likely to occur if the issuer’s information is high. Hence, lowering the face value is a more efficient way for the issuer to signal optimism about her cash flow.

The results of section 5 imply certain predictions that have been confirmed empirically. In particular, residential mortgage-backed security (RMBS) pools are typically divided into AAA-rated and mezzanine tranches, which are sold, and a junior “equity” tranche which is retained. Begley and Purnanandam (2016) find that when the equity tranche makes up a larger proportion of the loan pool’s face value, the loans in the RMBS pool have lower subsequent delinquency rates conditional on observables and the securities that are sold fetch higher prices conditional on their credit ratings.

To see how these are related to our results, let us interpret our cash flow as the aggregate repayment of a pool of mortgages. The proportion of the face value of the underlying mortgages that is included in the AAA-rated and mezzanine tranches then corresponds to the face value of our issuer’s security.\footnote{The division of the security into AAA-rated and mezzanine tranches likely results from regulations that require certain institutional investors to buy only investment-grade bonds. This institutional feature is not captured in our model.} In this context, our model predicts that a relatively larger equity tranche (which corresponds to a lower face value of the debt security in our model) signals to the market that the issuer expects a lower loan default
rate, which leads to a higher price of the securities that are sold. These predictions mirror the findings of Begley and Purnanandam (2016), described above.

Our main analysis assumes that the issuer’s type and the return of its portfolio are discretely distributed. In many applications, these distributions are continuous. We show in Section 6 that in the limit as types and portfolio returns become continuous, the optimal face value of debt is given by a simple differential equation that has a unique solution. Moreover, this equation describes an equilibrium of the continuous model, and the issuer's expected profits in the discrete model converge uniformly to her expected profits in the continuous model. Our solution for the continuous case has already proved to be a useful building block in several applied models.

In section 6 we show another result that further strengthens the model’s empirical plausibility. Suppose the issuer’s final cash flow is the aggregate repayment of a pool of loans, of which some known proportion are no-documentation loans. Let us interpret the issuer’s type as a local macroeconomic shock, which is distributed independently of the proportion of no-documentation loans in the pool. Plausibly, when the proportion of no-documentation loans is higher, the shock has a larger effect on loan repayment probabilities and thus on the issuer’s final cash flow. In this setting, we show that when an issuer has a higher proportion of no-documentation loans in her pool, she designs a security that has a lower face value, both conditional and unconditional on her information. Consistent with this prediction, Begley and Purnanandam (2016) find that issuers retain larger proportions of the face value of RMBS pools that contain a higher proportion of no-documentation loans.

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6 Manelli (1996, 1997) studies a general sequence of finite signaling games (which have finite type and message spaces) that converges to a continuous signaling game that, like ours, has compact type and message spaces. Manelli (1996) shows that any sequence of equilibria of the finite games has a convergent subsequence that converges to some equilibrium of the continuous game. Similarly, Manelli (1997) shows that if a sequence of equilibria of the finite games, each of which satisfies the Never a Weak Best Response criterion of Kohlberg and Mertens (1986), converges to some equilibrium of the continuous game, then this limiting equilibrium satisfies the same criterion. However, because Manelli’s finite games have finite message space while the discrete models in our Section 7 have an infinite message space (the set of monotone securities), his results cannot be directly applied to our setting.

7 In particular, see section 4.2 of DeMarzo (2005) and section 7.2 of Frankel and Jin (2015); the former cites an earlier version of our paper, DeMarzo (2003), for this result.
Other Related Literature

Our work is also related to He (2009), who considers the portfolio liquidation problem in a two asset generalization of Leland and Pyle (1979) with a risk-average issuer and risk-neutral investors. In that context, if the two assets’ returns are positively correlated conditional on the issuer’s information, then selling more of one asset lowers investors’ demand curve for the other. Intuitively, retaining more of the second asset is costlier, and thus a stronger quality signal, for a risk-averse issuer who keeps more of the first asset since asset returns are correlated. In our model, which instead assumes risk-neutrality, such cross-signaling also occurs but is due the correlation in the issuer’s information across assets. Also, the CARA-Normal framework in He (2009) precludes the consideration of securities whose payouts depend nonlinearly on underlying asset returns, which is a primary focus of our analysis.

Nachman and Noe (1994) consider the ex post security design problem of an issuer who needs to raise a fixed amount of capital to fund an investment opportunity. They also find that standard debt is optimal under certain assumptions.\(^8\) However, since the issuer’s target securitization revenue is fixed, there is pooling: the issuer sells a standard debt security whose face value does not depend on the issuer’s type. In our model, rather than having a fixed funding target, the issuer seeks to maximize its securitization profits. This yields a separating equilibrium in which the issuer signals high quality by borrowing less and issuing a more senior claim.

We focus throughout on a market setting in which the issuer can signal quality through quantity choices. Williams (2015) instead adapts the competitive search framework of Guerrieri, Shimer, and Wright (2010) to demonstrate that it is also possible to achieve a similar equilibrium outcome in which issuers signal quality by choosing the liquidity of the market in which they choose to trade, and shows that debt is the optimal monotone security in that context as well. An interesting extension might be to consider equilibrium liquidity choices when the issuer can sell multiple securities as in our paper.

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\(^8\) Their assumptions include Conditional Stochastic Dominance, which is equivalent to our Hazard Rate Ordering property (Nachman and Noe 1994, n. 13, p. 19), and the D1 refinement of Banks and Sobel (1987), which is stronger than the Intuitive Criterion.
Another possibility, in a dynamic context, might be to allow the issuer to signal by delaying its trades as in Varas (2014).

1. **The Asset Sale Game**

We first consider a setting in which a risk-neutral issuer holds a fixed portfolio of distinct assets. The issuer would like to raise cash by selling some or all of assets to a set of competitive risk-neutral investors. There are potential gains from trade as the issuer is less patient than the investors with regard to future cash flows. For instance, the issuer may have attractive alternative investments or face liquidity or capital requirements. There is also adverse selection: the issuer has private information about the future returns of its assets. In this section we formally define an equilibrium, construct a unique solution subject to a natural and weak refinement, and study an example.

1.1. **The Model and Equilibrium**

The model’s participants consist of a single issuer and a continuum of investors. All are risk-neutral and fully rational. The issuer holds an initial portfolio of $n$ assets represented by the row vector $a \in \mathbb{R}_{+}^{n}$, where $a_i > 0$ represents the number of shares held of asset $i$. Let $F_i$ denote the random future payoff of asset $i \in \{1, \ldots, n\}$. The issuer has private information about these payoffs, which is summarized by the issuer’s type $t \in \{0, \ldots, T\}$. Conditional on the type $t$, asset $i$ has an expected payoff $f_i(t) = E[F_i | t]$. Let $F \in \mathbb{R}^n$ be the column vector of random asset payoffs and let $f(t) = E[F | t]$ denote the column vector of expected payoffs conditional on $t$. We refer to $(a, f)$ as the issuer’s *endowment*.

Given its endowment, the issuer’s asset sale decision is represented by a row vector $q \in \mathbb{R}_{+}^{n}$, such that $0 \leq q \leq a$. That is, for each asset $i$, $q_i \leq a_i$ represents the number of shares of asset $i$ sold by the issuer. The issuer may also set a maximum price $p_i \in [0, \infty]$ for each asset $i$. If the market clearing price for asset $i$ exceeds $p_i$, the issuer charges $p_i$.

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9 For vectors $x$ and $y$, the notation “$x \leq y$” means that for all $i$, $x_i \leq y_i$. 


and rations the asset. In this way, the issuer can intentionally underprice the asset. Let $\overline{p}$ be the column vector of maximum prices.

After the issuer announces the asset sale decision $(q, \overline{p})$, each asset is sold to the investors at its market-clearing price unless the price cap set by the issuer is binding. Let $p_i$ denote the price received for asset $i$ and let $p \in \mathbb{R}_+^n$ be the column vector composed of these prices. Obviously, $p \leq \overline{p}$.

Investors have a common, positive prior over the different possible realizations of the issuer’s type $t$. On seeing the issuer’s sale decision $(q, \overline{p})$, the investors update this prior to some posterior beliefs $\mu(t \mid q, \overline{p})$. As investors are competitive and risk-neutral, their demand for asset $i$ is perfectly elastic at the price

$$\sum_{t} f(t) \mu(t \mid q, \overline{p}).$$

Given the vector $\overline{p}$ of maximum prices, the realized prices of the assets are thus given by the vector

$$p(q, \overline{p}) = \overline{p} \land \sum_{t} f(t) \mu(t \mid q, \overline{p}),$$

(1)

where $x \land y$ denotes the component-wise minimum of vectors $x$ and $y$.

The issuer discounts future cash flows at the rate $\delta \in (0,1)$ while investors have a higher discount factor that is normalized to one. The issuer may be less patient, e.g., because of capital requirements or access to worthwhile projects. This difference in discount factors is the source of the gains from trade in the model.

Given the quantity vector $q$ and price vector $p$, the issuer earns revenue $qp$ from the asset sale, plus the discounted expected payoff of the retained assets, for a total payoff:

$$U(t, q, p) = qp + \delta(a - q) E[F_t] = \delta a f(t) + q(p - \delta f(t)).$$

(2)

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10 We could also allow the issuer to set reserve or minimum prices for the securities. However, since investors would refuse to buy overpriced securities, extending the strategy space in this way would play no role in equilibrium. (In other auction environments, a reserve price is useful to extract additional surplus from buyers. Our model differs in that investors are homogeneous and uninformed. Hence they earn no surplus even absent a reserve price.)

11 As we will see, allowing for a price cap does not affect the existence of an equilibrium, but will prove useful in establishing uniqueness using only the Intuitive Criterion.
The following definition is standard.

**Asset Sale Equilibrium.** A *perfect Bayesian equilibrium* for the *asset sale game* is an issuance strategy \((q(t), \bar{p}(t))\) for the issuer, a price response function \(p(\cdot, \cdot)\), and a posterior belief function \(\mu(\cdot | q, \cdot)\) for the investors, such that the following conditions hold.

1. Payoff Maximization: for any type \(t\), the issuer’s sale decision \((q(t), \bar{p}(t))\) solves \(\max_{q', \bar{p}'} U(t, q', p(q', \bar{p}'))\) subject to \(0 \leq q' \leq a\).

2. Competitive Pricing: for any sale decision \((q, \bar{p})\), the price function satisfies equation (1).

3. Rational Updating: investors’ posterior beliefs \(\mu(t | q, \bar{p})\) are given by Bayes’ rule whenever possible (i.e., on the equilibrium path).

We also introduce the following natural terminology:

**Equilibrium Outcome.** The *equilibrium outcome* for the issuer is given by

\[
u(t) = q(t)\left[ p(q(t), \bar{p}(t)) - \delta f(t) \right].
\]

The outcome \(u(t)\) omits the fixed component \(\delta a f(t)\) of the issuer’s payoff in (2). Thus \(u(t)\) represents the additional surplus that an issuer of type \(t\) recovers through an asset sale. Further, we define a property that is a natural adaptation of the concept of a separating equilibrium to our context:

**Fair Pricing.** An equilibrium is *fairly priced* if, for all \(i\) and \(t\), \(q_i(t) > 0\) implies \(p_i(q(t), \bar{p}(t)) = f_i(t)\).

Roughly, a fairly priced equilibrium differs from a separating equilibrium because it requires only that no traded asset is mispriced.\(^{12}\) The lack of mispricing implies that investors’ payoffs are identically zero.

\(^{12}\) A separating equilibrium is not fairly priced if it involves underpricing. Conversely, a fairly priced equilibrium may not be separating if either (a) the mappings \(f_i\) are not invertible or (b) there are multiple types \(t\) that sell no securities. Since (a) and (b) can occur naturally in our context, we require fair pricing rather than separation.
1.2. Monotonicity and Uniqueness

In this section, we introduce a monotonicity assumption regarding the issuer’s information, and then construct a fairly priced equilibrium for the asset sale game. While as with all signaling games multiple equilibria are possible, we argue that this equilibrium is the unique one with “reasonable” beliefs. It is also the best fairly priced equilibrium for the issuer, and hence is Pareto optimal within the set of fairly priced equilibria (since investors’ payoffs are zero in all such equilibria).

We begin by assuming the issuer’s information can be ordered so that higher types have better news regarding the value of each asset:

**ASSUMPTION A (MONOTONE EXPECTED PAYOFFS).** For $t > s$, $f(t) \geq f(s)$. In addition, $f(0) \geq 0$ and $af(0) > 0$.

This assumption states that a higher value of $t$ is (weakly) good news regarding the expected payoff of each asset, that an asset’s expected payoff is never negative (e.g. because of limited liability), and that the portfolio has a positive value even with the worst possible news.\(^{13}\) If there is only one asset ($n = 1$), monotonicity is not restrictive since the types can be reordered. For $n > 1$, it implies that the ordering is common across the assets. Our leading example is a portfolio of securities backed by a common pool of assets, such as the debt and equity of a single firm or the mortgage-backed security tranches of a mortgage pool, with the issuer having private information about the future value of the underlying asset pool.\(^{14}\)

Using **ASSUMPTION A**, we demonstrate the existence of a fairly priced equilibrium by constructing the equilibrium inductively according to the following recursive linear program:

Given $u^*(s)$ for $s < t$, define:

$$u^*(t) = \max_q (1 - \delta) q f(t)$$

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\(^{13}\) Assumption A does not rule out zero (or even negative) realized payoffs for the securities, as long as the expected aggregate value of the portfolio remains positive.

\(^{14}\) In the case of asset-backed securities, a sufficient condition for the expected payoffs $f(t)$ to be nondecreasing in the issuer’s type $t$ is that (a) the realized payoffs $F$ are nondecreasing in the random future value $Y$ of the underlying asset pool and (b) the distribution of $Y$ is nondecreasing in $t$ in a first-order stochastic dominance sense.
subject to (i) $0 \leq q \leq a$ and (ii) for all $s < t$, $u^*(s) \geq q (f(t) - \delta f(s))$. \hspace{1cm} (3)

Let $q^*(t)$ be a solution to (3) for each $t$.\footnote{If the solution to (3) is multi-valued, then $q(t)$ can also be allowed to be a mixed strategy on the solution set. For convenience, we will often speak in terms of pure strategies, though all of the results hold in the general case.} Since constraint (ii) in (3) is vacuous for $t = 0$, it follows that $q^*(0) = a$ and $u^*(0) = (1 - \delta)af(0)$.

The intuition for the above problem is as follows. Suppose each type were to receive a fair price in equilibrium. Then upon issuing quantity $q$, the issuer would raise cash $qf(t)$ and hence reap the net surplus $(1 - \delta)qf(t)$. Each type maximizes its surplus subject to the constraint (ii) in (3) that no worse type would do better by mimicking it, where the payoff to type $s$ from mimicking type $t$ is the cash received $qf(t)$, less the true foregone discounted expected return $\delta qf(s)$ of the assets sold.

Constraint (ii) states only that worse types do not want to mimic better ones. Equilibrium also requires that better types do not want to mimic worse types. Including those constraints would not allow for a recursive solution, so we ignore them for now and show later (PROPOSITION 2) that they hold.

First, we show that a solution $u^*$ exists, which is nonincreasing.

**PROPOSITION 1.** The maximization problem given by (3) has a unique solution $u^*$, which is strictly positive and nonincreasing in the issuer’s type $t$.

**PROOF:** See Appendix. \hfill ✷

Why must $u^*$ be nonincreasing? The equilibrium payoff $u^*(s)$ of type $s$ clearly cannot be less than the payoff type $s$ would get by imitating type $t$ (i.e., by issuing $q^*(t)$ instead of $q^*(s)$). This imitation payoff, in turn, cannot be less than the equilibrium payoff $u^*(t)$ of type $t$ since the sales revenue $q^*(t)f(t)$ is the same but the opportunity cost of selling the assets is lower for type $s$: $\delta q^*(t)f(s) \leq \delta q^*(t)f(t)$ by ASSUMPTION A. Intuitively,
while type $t$ sells for a higher price, in order to separate from lower types it must raise less cash, reducing its attainable surplus.$^{16}$

Next we show that the unique solution to (3) can be supported as an equilibrium.

**Proposition 2.** There exists a fairly priced equilibrium of the asset sale game, $(q^*(\cdot), \bar{p}^*(\cdot), p^*(\cdot, \cdot), \mu^*(\cdot | \cdot, \cdot))$, with outcome $u^*(t) = (1-\delta)q^*(t)f(t)$. This equilibrium can be supported by off-equilibrium beliefs such that for all $q \in \mathcal{R}_+$ not chosen in equilibrium,

$$\mu^*(\tau^*(q) | q, \bar{p}) = 1$$

where $\tau^*(q)$ is the lowest type $t$ that minimizes $u^*(t) + \delta q f(t)$. The corresponding price function $p^*$ is increasing in the quantity vector $q$.

**Proof:** See Appendix. ♦

We prove this result by constructing an equilibrium in which each type $t$ issues the quantity vector $q^*(t)$ that solves (3) and the price vector $p^*$ is given by equation (1). In the equilibrium the issuer does not use rationing: any maximum price vector $\bar{p}^*(t) \geq f(t)$ may be chosen. The essence of the proof is to show that high types do not want to mimic low types, which we establish inductively using an approach related to that of Cho and Sobel (1990). Finally, we will motivate the particular choice of off-equilibrium beliefs (4) when we consider refinements below.

Perfect Bayesian equilibria are generally not unique in signaling games. Why focus on this particular equilibrium? First, it maximizes the issuer’s payoff and is thus efficient in the set of fairly priced equilibria:$^{18}$

**Proposition 3.** The equilibrium $(q^*(\cdot), \bar{p}^*(\cdot), p^*(\cdot, \cdot), \mu^*(\cdot | \cdot, \cdot))$ yields the highest payoff an issuer of each type $t$ can obtain in any fairly priced equilibrium.

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$^{16}$ This tradeoff of mispricing versus holding/retention costs is shared by many other models including Leland and Pyle (1977), Myers and Majluf (1984), DeMarzo and Duffie (1999), and He (2009). In all of these models, better types signal their quality by raising less external capital.

$^{17}$ That is, if $q$ is unexpected, beliefs are concentrated on $\tau^*(q) = \min\{\arg\min_t [u^*(t') + \delta q f(t')]\}$.

$^{18}$ As noted above, investors’ payoffs are identically zero in any fairly priced equilibrium. Thus, the efficient equilibrium is the one that maximizes the issuer’s payoff.
**PROOF:** In any fairly priced equilibrium \((q, \bar{p}, p, \mu)\), the payoff of an issuer of type \(t\) is \(u(t) = (1-\delta)q(t)f(t)\). We proceed by induction. In the base case \(t = 0\), \(q(0) = a\) by (2), so \(u(0) \leq u^*(0)\). Now consider \(t > 0\), and suppose \(u(s) \leq u^*(s)\) for all \(s < t\). The IC constraint for each type \(s < t\) in the candidate equilibrium is

\[
q(t)[f(t) - \delta f(s)] \leq u(s) \leq u^*(s).
\]

But then \(q\) is feasible in (3) for type \(t\), so \(u(t) \leq u^*(t)\) by (2). \(\diamondsuit\)

We next show that \(u^*\) is the unique equilibrium outcome when we restrict investors to reasonable beliefs following a deviation by the issuer. Of the belief refinements introduced in the literature, the weakest is the Intuitive Criterion of Cho and Kreps (1987). In our context, this refinement simply states that if investors see an out-of-equilibrium sale decision \((q, \bar{p})\), their beliefs should put weight only on those types who could possibly expect to gain from the deviation.

In the context of the asset sale game, suppose type \(s\) makes the deviation \((q, \bar{p})\). Type \(s\) must lose from this deviation if its equilibrium payoff \(u(s)\) exceeds its maximum deviation payoff \(q[\bar{p} - \delta f(s)]\) or, equivalently, if \(q\bar{p} < u(s) + \delta qf(s)\). The Intuitive Criterion states that investors should assign probability zero to the issuer’s type being \(s\), as long as some other type \(t\) might gain from the deviation:

**THE INTUITIVE CRITERION.** A perfect Bayesian equilibrium \((q(\cdot), \bar{p}(\cdot), p(\cdot, \cdot), \mu(\cdot | \cdot, \cdot))\) of the asset sale game, with outcome \(u(\cdot)\), is **intuitive** if, for any vectors \(\bar{p}, q \in \mathbb{R}_n^a\) such that (i) \(0 \leq q \leq a\) and (ii) \(q\bar{p} \geq u(t) + \delta qf(t)\) for some type \(t\),

\[
q\bar{p} < u(s) + \delta qf(s) \text{ implies } \mu(s | q, \bar{p}) = 0.
\]

For brevity, we will call investors’ beliefs **intuitive** if they satisfy the Intuitive Criterion. We now show that the equilibrium outcome in **PROPOSITION 2** is the unique outcome satisfying this requirement: 19

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19 Other refinements in the signaling literature (such as D1, NWBR, and strategic stability) are stronger than the Intuitive Criterion, and the outcome \(u^*\) survives them as well. The reason is that an equilibrium exists
**PROPOSITION 4.** Investors’ beliefs $\mu^*$ in the equilibrium identified in PROPOSITION 2 are intuitive. Moreover, every intuitive equilibrium of the asset sale game is fairly priced, has the same outcome $u = u^*$, and has an asset sale function $q^*(\cdot)$ that solves (3).

**PROOF:** See Appendix. 

The beliefs in (4) guarantee that when investors observe an unanticipated sale decision $(q, \overline{p})$, they believe the issuer’s type is the lowest among those that have the strongest incentive to choose the given deviation. These beliefs clearly satisfy the Intuitive Criterion.

To see why every intuitive equilibrium is fairly priced, suppose in equilibrium type $t$ sells quantity $q$ and receives price $p$ such that some asset $i$ is underpriced: $q_i > 0$ and $p_i < f(t)$. Suppose type $t$ reduces the quantity of asset $i$ to some $q_i' < q_i$ for which

$$q_i'[f_i(t) - \delta f_i(t)] = q_i[p_i - \delta f_i(t)].$$

This equality means that if investors respond by assigning the price $f_i(t)$ to asset $i$, the issuer’s payoff from selling asset $i$ is unchanged. But the payoff of any lower type $s$ for which $f_i(s) < f_i(t)$ would be reduced by this deviation:

$$q_i'[f_i(t) - \delta f_i(s)] < q_i[p_i - \delta f_i(s)],$$

since type $s$ suffers the decrease in quantity for a larger surplus. Because the deviation to $q_i'$ and price cap $\overline{p}_i = f_i(t)$ is unprofitable except for types $s$ with $f_i(s) \geq f_i(t)$, investors with intuitive beliefs must be willing to pay at least $f_i(t)$ for asset $i$; thus underpricing cannot occur in an intuitive equilibrium. Finally, because investors are rational and cannot have negative expected payoffs, overpricing cannot occur either, and so every intuitive equilibrium must be fairly priced.

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20 This intuition ignores what happens to securities $j \neq i$. A rigorous treatment appears in the appendix.
1.3. A Numerical Example

We have shown thus far that given a set of assets satisfying ASSUMPTION A, the equilibrium can be characterized as the solution to the recursive linear programming problem (3), which can be easily solved numerically. We conclude this section with a brief example:

**Example 1.** Suppose the issuer holds 1 share of each of 2 assets, with expected payoffs of each (conditional on $t = 0, 1, \ldots, 200$) given in Figure 1.

![Figure 1: Expected Payoffs of Assets for Example 1.](image)

Asset 1’s returns are less sensitive to information than asset 2’s returns for low $t$, but are more sensitive for high $t$.

While the overall sensitivity to information for each asset is similar (a 10% increase in value over the range of $t$), the right panel shows that asset 2’s return is more sensitive to $t$ for $t < 50$ and asset 1 is more sensitive (locally) for $t > 50$. How will this difference in informational sensitivity affect the liquidity of each security in equilibrium? And how is the price of each security affected by the quantity of the other security that is sold?

Figure 2 depicts the equilibrium strategies for $\delta = 0.9$ (given by the solid line with circles for $t = 0, 10, \ldots, 200$). For $t \leq 50$, the issuer signals high quality by retaining more of asset 2 while selling all of asset 1. For $t > 50$, the qualitative nature of the solution changes abruptly: the issuer signals high quality by retaining more of asset 1 and issuing more of asset 2. Finally, for the highest levels of $t$, asset 1 is completely retained, and
high quality is again signaled by retaining asset 2. Intuitively, asset 1 is “more liquid” when $t \leq 50$ and its sensitivity to information is relatively low, whereas the reverse is true when $t$ is high and asset 1 becomes more informationally sensitive than asset 2.

Also depicted in Figure 2 is the equilibrium price function, as a contour plot showing the type investors infer given any quantity $q$; darker shades indicate lower types $t$, while white lines indicate the iso-price contours for $t = 0, 10, \ldots 200$. On each such contour, investors’ posterior beliefs – given by equation (4) – are constant, and thus the prices of both assets are constant. As the contours are downward sloping, beliefs decline as the quantity sold of either asset increases. Moreover, for types $t$ between 50 and 120, beliefs drop discontinuously for a small increase in quantity. For example, starting from $q^*(60)$, a small increase in the quantity sold of either security will cause beliefs to drop discontinuously from 60 to below 20; similarly, a small increase from $q^*(90)$ would cause beliefs to drop to 0. This discontinuity occurs because in the optimization (3), the binding incentive constraint is non-local; for example, the type with the greatest incentive to mimic type 60 is type 18. The possibility that non-local constraints may bind distinguishes our setting from standard signaling models in which a single crossing property holds.
In Section 3, we generalize and formalize the above intuitions regarding how the relative informational sensitivities of an issuer’s assets affect the optimal issuance strategy. But first, we consider the advantages of selling securities separately versus as a pool, and the implications for ex ante security design.

2. Ex ante Security Design: Splitting Securities

In section 1, we took the issuer’s assets as exogenous. In many applications this assumption is entirely appropriate: the issuer may only have the option of selling existing assets from its portfolio. But in some circumstances, the issuer can split the future cash flow of a given asset among several securities. For example, investment banks holding mortgage pools often split these pools into distinct tranches prior to selling them to investors. Similarly, a corporation decides how to apportion the future cash flows of its assets among different securities, such as debt and equity. In this section we begin to
consider the implications of the preceding analysis for optimal security design, and show that the issuer benefits by splitting existing assets into smaller sub-pieces prior to observing its type. Splitting assets gives the issuer additional flexibility in the issuance decision, allowing better types to signal their quality more efficiently. This result may explain, for example, some of the gains associated with tranching mortgage pools into collateralized mortgage obligations (CMOs).\textsuperscript{21}

We first prove the following general result and then consider several examples:

**PROPOSITION 5.** Consider two endowments, \((\hat{a}, \hat{f})\) and \((a, f)\), satisfying ASSUMPTION A. Let \(q^*\) be the equilibrium sale decision given \((a, f)\), and suppose that for each \(t\) there exists a non-negative portfolio \(\hat{q}(t) \leq \hat{a}\) with the same conditional payoffs: \(\hat{q}(t) \hat{f} = q^*(t)f\). Then for every type, the issuer’s equilibrium payoff given \((\hat{a}, \hat{f})\) equals or exceeds the payoff given \((a, f)\).

**PROOF:** The proof then follows by induction on the type \(t\). The base case is trivial:

\[
\hat{u}^*(0) = (1-\delta)\hat{a}f(0) \geq (1-\delta)q(0)\hat{f}(0) = (1-\delta)q^*(0)f(0) = (1-\delta)af(0).
\]

Now let \(t > 0\) and suppose that for all \(s < t\), \(\hat{u}^*(s) \geq u^*(s)\). Consider the problem (3) for \(\hat{u}(t)\). Because \(\hat{q}(t) \hat{f} = q^*(t)f\),

\[
\hat{q}(t)[\hat{f}(t) - \delta\hat{f}(s)] = q^*(t)[f(t) - \delta f(s)] \leq u^*(s) \leq \hat{u}^*(s).
\]

Thus, \(\hat{q}\) is feasible for type \(t\). Hence, \(\hat{u}^*(t) \geq (1-\delta)\hat{q} \hat{f}(t) = (1-\delta)q^*(t)f(t) = u^*(t)\).

The intuition for **PROPOSITION 5** is simple. Whatever portfolio \(q^*(t)\) that type \(t\) issues given \(f\), that same payoff can be issued with \(\hat{f}\) by issuing \(\hat{q}(t)\), and hence the constrained maximization problem in (3) cannot yield the issuer a lower payoff with \((\hat{a}, \hat{f})\) than with \((a, f)\).

\textsuperscript{21} It is not automatic that increased flexibility benefits the issuer, since in a strategic setting there can be gains to commitment. Indeed, in this setting the seller could gain by committing ex ante to quantities (e.g., committing to sell all assets before any information is learned). Our results here show that while committing to quantities may be helpful, committing to ratios (of the quantity of one asset to another) is not.
As an immediate application, suppose that before learning its information \( t \), the issuer has the opportunity to split all or some of the assets into tranches. For example, it might be possible to split asset \( F_1 \) into securities \( F_{1a} \) and \( F_{1b} \) such that \( F_1 = F_{1a} + F_{1b} \). Because any portfolio involving \( F_i \) has the same conditional payoff as a portfolio with equal quantities of \( F_{1a} \) and \( F_{1b} \), PROPOSITION 5 immediately implies that this splitting cannot harm the issuer.

**Example 2.** Consider the case of the issuer with two securities in Example 1. Suppose the issuer instead only had a single pooled asset with conditional payoffs \( f = f_1 + f_2 \). By PROPOSITION 5, an issuer selling this pooled asset cannot do better than an issuer with the two separate securities. In particular, an issuer with a single asset is forced to issue \( f_1 \) and \( f_2 \) in equal proportions, whereas an issuer with the two securities finds it optimal to vary the proportions as shown in Example 1. The lower two curves in Figure 3 illustrate the issuer’s payoffs in each case (the other two curves will be discussed below).

PROPOSITION 5 implies that absent transaction costs, there is no limit to the splitting that should occur.\(^{22}\) In Section 5 we consider the possibility of “unlimited” splitting – which generates the highest possible issuer payoff shown in Figure 3 – and show that it is equivalent to designing a single security after the issuer learns its private information.

In practice, even when securities are designed ex ante, there may be limits to the number of securities that can be issued. In that case there remains the question of how these securities should be designed. A further implication of PROPOSITION 5 is that optimal securities should not be “interior” in the space of feasible designs. To see why, consider two securities \((F_1, F_2)\) and without loss of generality suppose there is a quantity one of each. Suppose \( F_1 \) is interior, in the sense that \( F_1 - \varepsilon F_2 \) is a feasible design for sufficiently small \( \varepsilon > 0 \). Then the issuer could gain by using the following alternative feasible design:

\[
(\hat{F}_1 = F_1 - \varepsilon F_2, \hat{F}_2 = [1 + \varepsilon] F_2).
\]

\(^{22}\) DeMarzo and Duffie (1999) consider the case in which the issuer chooses a single security (tranched from a larger pool) to issue prior to learning the information \( t \). The results here show that the issuer can do even better by splitting the cash flows further, creating multiple tranches to be sold.
Because any feasible portfolio of \((F_1, F_2)\) can be replicated as a portfolio of \((\hat{F}_1, \hat{F}_2)\), but not vice versa,\(^{23}\) by \textsc{Proposition} 5 the issuer cannot be harmed (and generically will gain) by holding securities \((\hat{F}_1, \hat{F}_2)\) in place of \((F_1, F_2)\).

In the space of monotone security designs, a firm’s debt and levered equity securities are commonly used “extreme” securities:\(^{24}\) the payoff to levered equity in low states cannot reduced without violating limited liability, and the payoff to debt cannot be reduced in high states without violating monotonicity. In the setting of \textsc{Example} 1, consider pooling the original two securities as in \textsc{Example} 2, but then tranching the pool into a debt and equity security. Figure 3 shows the issuer’s payoff from doing so when the debt has a face value of 172 given the pool’s payoff (conditional on the type \(t\)) is normal with a variance of 40.\(^{25}\) As shown in Figure 3, the issuer’s profits from pooling the original assets and tranching them into a senior debt security and a junior equity tranche are even higher than from selling them individually.

\(^{23}\) For any \(c_1, c_2 \in [0,1]\), the portfolio \(c_1F_1 + c_2F_2\) is equivalent to the portfolio \(c_1\hat{F}_1 + \frac{c_2 + c_1E}{1 + \varepsilon} \hat{F}_2\), which is feasible as both assets weights lie in \([0,1]\). In contrast, the portfolio \(c_1\hat{F}_1 + c_2\hat{F}_2\) is equivalent to \(c_1F_1 + [c_2(1+\varepsilon) - c_1\varepsilon]F_2\), which is infeasible if \(c_2\) lies in the nonempty interval \(\left[0, c_1\varepsilon/(1 - \varepsilon)\right)\).

\(^{24}\) By extreme we mean that the conditional payoffs cannot be replicated by a positive convex combination of other feasible securities.

\(^{25}\) It is now necessary to specify this conditional distribution (which we did not need in order to compute the prior equilibria) as the payoff of the debt and equity tranches are non-linear in the cash flows of the pool. The result that pooling and tranching into a debt-equity split is superior to individual sales does not depend on this distributional choice.
Debt and equity securities naturally differ in their priority and, consequently, in their informational sensitivity. In the following sections we give a formal definition of differential informational sensitivity, see how it can arise naturally as a result of priority versus subordination, and study the consequences for optimal asset sales and security design.

3. Informationally Ordered Assets

In this section we return to the case of a fixed set of assets considered in section 1. We introduce an ordering of these assets based on the sensitivity of their returns to the issuer’s information. For assets that can be so ordered, the equilibrium of the asset sale game has a simple form: the issuer sells its least informationally sensitive assets first.
This result formalizes the intuition that less informationally sensitive securities should be more liquid.

3.1. Information Sensitivity

We say asset $i$ is more informationally sensitive than asset $j$ if its expected value changes by a larger percentage for a given change in the issuer’s type $t$. To ensure percentage changes are well-defined, we assume:

**Assumption B (Positive Expected Payoffs).** For all $i, f_i(0) > 0$.\(^{26}\)

This permits the following definition:

**Informational Sensitivity.** Asset $i$ is more informationally sensitive at $t$ than asset $j$ if, for all $s < t$, $f_i(t)/f_i(s) > f_j(t)/f_j(s)$.\(^{27}\) Asset $i$ is more informationally sensitive than asset $j$ if the above holds for all $t$. The assets display increasing information sensitivity (IIS) if for all $i > j$, asset $i$ is more informationally sensitive than asset $j$.\(^{28}\)

Intuitively, one would expect that the asymmetric information problem would be least severe for those assets that are least sensitive to the issuer’s private information. Hence, we might expect these assets to create the least price impact when sold, and thus be the most attractive to sell first. This intuition can be made precise as follows.

**Proposition 6.** Suppose asset $i$ is more informationally sensitive at $t$ than asset $j$. Then for the equilibrium $q^*$, if $q^*_i(t) > 0$ then $q^*_j(t) = a_j$: a type $t$ issuer will not sell any of asset $i$ unless it sells asset $j$ in its entirety.

**Proof:** Omitting the constraint $0 \leq q \leq a$ in (3) and terms that are independent of $q$, the Lagrangian is $qf(t) - \sum_{s<t} \lambda(s)q[f(t) - \delta f(s)]$. The derivative with respect to $q_i$ can then be written as $\delta \sum_{s<t} [\lambda(s)f_i(s)] - kf_i(t)$, where $k = [\sum_{s<t} \lambda(s)] - 1$. Thus, $q^*_i(t) > 0$ implies

\(^{26}\) Under this requirement, a security can have zero payoffs in some states as long as, for any type $t$, there are states in which the security’s payoff is positive.

\(^{27}\) This condition is equivalent to $f_i(t)/f_i(s)$ increasing in $t$, or alternatively, $\frac{d}{dt} \ln f_i > \frac{d}{dt} \ln f_j$.

\(^{28}\) IIS is equivalent to the functions $f_i(t)$ being log-supermodular in $(i, t)$.
\[ \sum_{s \in \mathcal{I}} [\lambda(s)f(s)/f(t)] \geq k/\delta. \] Since \( f(s)/f(t) < f_j(s)/f_j(t) \), it follows that \( \sum_{s \in \mathcal{I}} \lambda(s)f(s)/f(t) > k/\delta \), and thus \( q_j(t) = a_j \).

### 3.2. Hurdle Class Strategies

A further characterization of equilibrium is possible under IIS when all of the issuer’s assets can be ordered according to informational sensitivity. In this case, **Proposition 6** immediately implies that the issuer will choose to sell all of its less informationally sensitive assets and retain all of its more informationally sensitive ones, with the exact cutoff, or hurdle class, determined by the issuer’s type. Moreover, as is shown below, this hurdle class is decreasing in the issuer’s type.

**Proposition 7.** Suppose the issuer’s assets display Increasing Information Sensitivity. Then for each type \( t \), there exists a cutoff or hurdle class \( c(t) \) such that \( q_i^*(t) = a_i \) for \( i < c(t) \) and \( q_i^*(t) = 0 \) for \( i > c(t) \). Furthermore, for \( t > s \), \( q^*(t) \leq q^*(s) \) and \( c(t) \leq c(s) \).

**Proof:** The existence of a hurdle class \( c(t) \) follows as an immediate corollary of IIS and **Proposition 6**. The existence of a hurdle class implies that \( q^* \) is ordered; that is, either \( q^*(t) \leq q^*(s) \) or \( q^*(s) \leq q^*(t) \). Also recall from **Proposition 1** that \( u^*(t) = (1-\delta)q^*(t)f(t) \leq u^*(s) = (1-\delta)q^*(s)f(s) \). Since \( f(t) \geq f(s) \) by **Assumption A**, it must be that \( q^*(t) \leq q^*(s) \): \( q^* \) is decreasing. Hence, \( c(t) \leq c(s) \) as well.

This result implies that the optimal issuance decision \( q^*(t) \) can be restricted to the set \( C = \{ q \mid \text{for some integer } c, q_i = a_i \text{ for } i < c \text{ and } q_i = 0 \text{ for } i > c \} \) of quantities having a cutoff or hurdle class. In other words, the issuer’s decision is reduced to a one-dimensional quantity choice. Moreover, as we show next, the local incentive compatibility constraint in (3) binds, allowing us to characterize the equilibrium via a simple difference equation:

**Proposition 8.** If the assets have increasing information sensitivity, then \( q^*(0) = a \), and for \( t > 0 \), \( q^*(t) \) is the unique element of \( C \) such that

\[ q^*(t)[f(t) - \delta f(t-1)] = (1-\delta)q^*(t-1)f(t-1). \tag{5} \]

**Proof:** See Appendix.
Thus, in the IIS case, equilibria can be easily characterized. Even though the signal space available to the issuer is multi-dimensional, the game essentially collapses into a one-dimensional signaling game for which standard solution techniques are available. In the next section, we demonstrate an application in which IIS arises naturally.

4. An Application: Prioritized Securities

We showed in section 3 that if assets can be ordered according to their informational sensitivity, then the asset sale game has a simple solution. We now show that in settings in which an issuer possesses securities that are backed by a common (or equivalent) asset pool, more senior securities will have lower informational sensitivity under a natural distributional assumption, the Hazard Rate Ordering.

Specifically, suppose the issuer possesses securities that are backed by pool of underlying assets with aggregate stochastic cash flow $Y \in \mathbb{R}_+$. The payoff of security $i \in \{1, \ldots, n\}$ is given by $F_i(Y)$ with $F_i$ nondecreasing and nonnegative. For convenience, we write $Y_i$ to represent the aggregate cash flow conditional on the issuer’s type $t$, so that the conditional expected payoff of asset $i$ is

$$f_i(t) = E[F_i(Y) \mid t] = E[F_i(Y_i)].$$

We continue to assume ASSUMPTION A and ASSUMPTION B, which require that $f_i(t)$ is positive and nondecreasing. While first order stochastic dominance is sufficient to assure this monotonicity, in order to obtain an even sharper characterization of the issuer’s behavior we make the following somewhat stronger assumption:

**Hazard Rate Ordering (HRO).** For all $t$, $Y_i$ has a common support; and for all $t > s$, $\Pr(Y_i \geq y) / \Pr(Y_s \geq y)$ is decreasing in $y$, for $y$ in the support of $Y$.

This property is weaker than the Monotone Likelihood Ratio Property (MLRP), which is commonly assumed in signaling environments.

29 At the upper boundary of $Y$’s support, $\Pr(Y \geq y \mid t)$ may be zero; we only require the ratio be increasing up to this point. This definition is equivalent to $g(y \mid t) / [1 - G(y \mid t)]$ decreasing in $t$ where $g$ (resp., $G$) is the conditional density (distribution) of $Y$, but it applies also if $Y$ has atoms or is discrete.

30 The MLRP states that the ratio of the conditional densities $g(y \mid s) / g(y \mid t)$ increases with $y$ for $t > s$. This property implies HRO, which in turn implies FOSD (see, e.g., Ross (1983)).
In order to rank securities, we need to define a notion of seniority. Intuitively, a more senior security should receive a greater share of “early” cash flows than a more junior one. We can formalize this intuition as follows.

**SENIORITY AND PRIORITIZED SECURITIES.** Security \( j \) is junior to security \( i \), or equivalently \( i \) is senior to \( j \), if \( F_j(y) / F_i(y) \) is nondecreasing and nondegenerate for \( y \) in the support of \( Y \).\(^{31}\) The set of securities \( \{F_i\} \) is prioritized if \( j > i \) implies \( j \) is junior to \( i \).

Roughly speaking, a more senior security is paid a relatively greater share of early cash flows while a more junior security receives more of the later cash flows. If \( F_j \) is junior to \( F_i \), then its payoff can only cross that of \( F_i \) from below; the same is true of any positive scalar multiples of \( F_j \) and \( F_i \). We could also state the condition in terms of the sensitivity of final returns: the return of security \( j \) is more sensitive to \( Y \) than is that of security \( i \).

This definition generalizes standard notions of seniority. For example, senior debt is clearly senior to junior debt or levered equity: its payoff becomes constant before the more junior securities are paid. It is also senior to *unlevered* equity; while they are both paid up front, equity receives a larger share of higher cash flows. Figure 4 depicts some further examples. In the figure, the securities are ranked in order, with \( F_1 \) the most senior and \( F_5 \) the most junior, with the only exception that \( F_4 \) is noncomparable with \( F_2 \) or \( F_3 \). Thus \( (F_1,F_4,F_5) \) and \( (F_1,F_2,F_3,F_5) \) are prioritized sets.

\(^{31}\) If \( F_i(y) = 0 \), this condition requires \( F_j(y) = 0 \), and we interpret the ratio at such points as equal to \( \inf \{ F_j(y) / F_i(y) \colon y \in \text{supp}(Y), F_i(y) > 0 \} \). By nondegenerate we mean that the ratio, as a function of \( Y \), takes on more than one value with a positive probability.
We can now state the following important result, relating a security’s seniority to the conditions under which it will be liquidated: under HRO, if one security is junior to another, then it will be more informationally sensitive, and so the former security will be sold only after the latter has been completely liquidated.

**PROPOSITION 9.** Suppose HRO holds and the securities are prioritized. Then IIS holds, and the issuer will not sell any portion of a given security unless it also sells its more senior securities in their entirety.

**PROOF:** See Appendix.

As an immediate application of this result, if we consider a standard, strictly prioritized capital structure consisting of senior debt, junior debt, and equity claims, then PROPOSITION 9 implies that the quality of the issuer’s information will be perfectly correlated with seniority of the securities it issues: while all types will sell senior debt claims, the best issuers will refrain from selling junior debt, and only the worst types will sell equity securities.

The results of PROPOSITION 9 are applicable even if the underlying assets differ across securities, as long as the asset payoffs have equivalent distributions. For example, issuers
of mortgage-backed securities may hold tranches of different mortgage pools, but as long as the pools have similar conditional distributions with respect to the issuer’s information (for example, with respect to prepayment risk), we should expect the issuer first to sell the more senior tranches of its various mortgage pools.  

5. Security Design

We now apply the tools we developed in prior sections to the problem of security design. We generalize the single ex-ante security case studied by DeMarzo and Duffie (1999) by permitting an issuer to design multiple monotone securities before and/or after she receives her information. Our main result is that under the Hazard Rate Ordering property, the issuer optimally designs a single standard debt security \( \text{ex post} \), with a face value that is declining in her type. Or she can maximally tranche her cash flow into prioritized securities \( \text{ex ante} \) and then selling those whose seniority exceeds a type-dependent threshold as studied in section 4. These approaches are equivalent: the portfolios sold by the issuer under the two approaches pay out the same aggregate cash flow to investors and fetch the same price. We also show how to compute the issuer’s optimal face value and seniority threshold as functions of the issuer’s type.

5.1. The General Security Design Game

Consider an issuer with a given asset portfolio with an unknown future cash flow \( Y \). She wishes to design securities to maximize her securitization profits. She can design any number of monotone securities, both before and after she sees her information. Formally, let \( Y \) be the support of the cash flow \( Y \). The General Security Design (GSD) Game is as follows.

\textbf{GSD GAME.} The issuer first partitions her cash flow into a finite set \( A = \{ A_i \}_{i=1}^K \) of \textit{ex ante} securities \( A_i : Y \rightarrow \mathbb{R}_+ \). She then sees her type \( t \) and offers investors a finite vector \( P_t = \{ P_t^j \}_{j=1}^K \) of \textit{ex-post} securities \( P_t^j : [0, \bar{Y}]^K \rightarrow \mathbb{R}_+ \) as well as a

\[ 32 \text{ Even if the conditional distributions are not completely identical, the result will hold approximately in the sense that securities with very different seniority are still likely to be ranked by their information sensitivity even if the underlying assets differ.} \]
vector \( \rho_t = \left( \rho_t^j \right)_{j=1}^{L_t} \in \mathbb{R}_{+}^{L_t} \) of price caps for these \textit{ex-post} securities. (The payout of \textit{ex-post} security \( j \) is a function \( P_t^j(A(Y)) \) of the vector \( A(Y) \) of \textit{ex-ante} security payouts.) Investors then assign a price \( p^j_t \) to each \textit{ex-post} security \( j = 1, \ldots, L_t \). Let \( W_P^A(Y) = \sum_{j=1}^{n(p)} P^j(A(Y)) \) denote the aggregate payout to investors that results from the \textit{ex-ante} and \textit{ex-post} security vectors \( A \) and \( P \).\(^{33}\) The payoff of a type-\( t \) issuer equals her fixed discounted expected cash flow \( \delta E[Y \mid t] \) (which can be ignored) plus her securitization profits \( \left( \sum_{j=1}^{L_t} p^j_t \right) - \delta E[W_P^A(Y) \mid t] \).

This definition includes many securitization schemes observed in practice. For instance, the issuer may design, \textit{ex ante}, a senior tranche \( A^1(y) = \min \{ c, y \} \) with face value \( c \) and a mezzanine tranche \( A^2(y) = \min \{ c', y - A^1(y) \} \) with face value \( c' \). After discovering her type \( t \), she may then sell a quantity \( z^i_t \) of each tranche \( i = 1, 2 \). Formally, her two \textit{ex post} securities are then given by \( P_t^1(A(y)) = z^i_t A^i(y) \) for \( i = 1, 2 \). This is a two-security generalization of DeMarzo and Duffie (1999). Alternatively, the issuer may sell senior and mezzanine debt \textit{ex post} with type-contingent face values \( c_i \) and \( c'_i \) respectively. Formally, she designs a single \textit{ex-ante} pass-through security \( A^1(y) = y \), while her two \textit{ex-post} securities are then given by \( P_t^1(A(y)) = \min \{ c_i, y \} \) and \( P_t^2(A(y)) = \min \{ c_i', y - P_t^1(A(y)) \} \). This is a two-security generalization of the one-security case used by DeMarzo (2005) and Frankel and Jin (2015).

Let \( g(t) \) denote the \textit{ex-ante} probability that the issuer is of type \( t \). Let \( \#(V) \) denote the dimension (number of components) of a vector \( V \). We define equilibrium in the GSD game as follows.

\textbf{EQUILIBRIUM: GSD GAME.} A perfect Bayesian equilibrium of the General Security Design game consists of an \textit{ex-ante} security vector \( A \) and, for each type \( t \),

\(^{33}\) The notation \( \#(V) \) denotes the dimension (number of components) of a vector \( V \).
an ex-post security decision \((P, \rho_t)\), together with a price function 
\[ p(A, P, \rho) = \left( p^i(A, P, \rho) \right)_{j=1}^{\#(P)} \]
and a belief function \(\mu(t \mid A, P, \rho)\), such that the following conditions hold.

1. Payoff Maximization: (a) the ex-ante choice \(A\) of the issuer lies in 
\[ \arg \max_A \left\{ \sum_t g(t) \left[ \sum_{t=1}^{L_t} p^i(A', P, \rho_t) - \delta E \left[ W_i^A(Y) \mid t \right] \right] \right\} \]
and (b) for each \(t\), the ex-post choice \((P_t, \rho_t)\) of a type-\(t\) issuer lies in 
\[ \arg \max_{(P_t, \rho_t)} \left\{ \sum_{t=1}^{L_t} p^i(A, P, \rho) - \delta E \left[ W_i^A(Y) \mid t \right] \right\} . \]

2. Competitive Pricing: for any choice \((A, P, \rho)\) of the issuer, the price function 
\[ p(A, P, \rho) = \rho \wedge \sum_t E[P(A(Y))] \mu(t \mid A, P, \rho) . \]

3. Rational Updating: for any choice \((A, P, \rho)\) of the issuer, the investors’ belief function \(\mu(t \mid A, P, \rho)\) follows Bayes’s Rule if possible.

The GSD game’s outcome is the function 
\[ u(t) = \left( \sum_{j=1}^{L_t} p^i(A, P, \rho_t) \right) - \delta E \left[ W_i^A(Y) \mid t \right] \]
securitization payoffs of each type \(t\). The definitions of fair pricing and the Intuitive Criterion in GSD games are as follows.

**FAIR PRICING (GSD GAME).** An equilibrium \(\left( A, (P_t, \rho_t)_{t=0}^T, p, \mu \right)\) of the GSD game is fairly priced if, for each type \(t\) and each ex-post security \(j = 1, \ldots, L_t\), the price of ex-post asset \(j\) equals its expected value conditional on the issuer’s type:
\[ p^i(A, P, \rho_t) = E \left[ P_i^j(A(Y)) \mid t \right] . \]

**INTUITIVE CRITERION (GSD GAME).** A fairly priced equilibrium \(\left( A, (P_t, \rho_t)_{t=0}^T, p, \mu \right)\) of the general security design game is intuitive if, for any

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35 In the asset sale game, we require fair pricing of an asset only if the issuer offers some of it for sale. In the GSD game each ex-post asset must be priced fairly since, by assumption, the issuer sells it in its entirety.
security choice \( (A', P', \rho') \) whose maximum attainable revenue \( \sum_j \rho_j^{t'} \) is at least as high as the minimum revenue \( u(t) + \delta E \left[ W_{P'} (Y) \right] \) that some type \( t \) requires to deviate, the posterior belief \( \mu(s \mid A', P', \rho') \) is zero for any type \( s \) for which the maximum attainable revenue \( \sum_j \rho_j^{t'} \) is strictly less than the minimum revenue \( u(s) + \delta E \left[ W_{P'} (Y) \right] \) that type \( s \) requires to deviate.

GSD games have complex strategy spaces. Fortunately, we can restrict the issuer to a much simpler set of strategies without altering the set of intuitive, fairly priced outcomes:

**Proposition 10.** Let \( E = \left( A, (P_t, \rho_t)_{t=0}^T, \rho, \mu \right) \) be any fairly priced equilibrium of the GSD game with outcome \( u(\cdot) \). Then the following profile \( \tilde{E} = (\tilde{A}, (\tilde{P}_t, \tilde{\rho}_t)_{t=0}^T, \tilde{\rho}, \tilde{\mu}) \) is a fairly priced equilibrium of the GSD game, with the same outcome \( u(\cdot) \). Moreover, if \( E \) is intuitive, then so is \( \tilde{E} \).

1. **Securitization.** In \( \tilde{E} \), the issuer first designs a single *ex-ante* pass-through security: \( \tilde{A}^1 (Y) = Y \). Each type \( t \) then issues a single *ex-post* security, whose payout \( \tilde{P}_t^1 (Y) = \tilde{P}_t^1 (\tilde{A} (Y)) \) equals the sum \( \sum_{j=0}^{l_t} P_j^t (A(Y)) \) of type \( t \)'s *ex-post* security payouts in \( E \). She can choose any price cap \( \tilde{\rho}_t^1 \) as long as it is not less than the security’s conditional expected payout \( E \left[ \tilde{P}_t^1 (Y) \mid t \right] \).

2. **Pricing.** For any action \( a' = (A', P', \rho') \) of the issuer, the price vector \( \tilde{p} (a') \) in \( \tilde{E} \) equals \( \rho' \wedge \sum_j \left[ E[P' (A' (Y))] \mid t \right] \tilde{\mu}(t \mid a') \).

3. **Beliefs.** Beliefs \( \tilde{\mu} \) in \( \tilde{E} \) follow Bayes’s Rule whenever possible. If instead the issuer’s action \( a' = (A', P', \rho') \) is unexpected in \( \tilde{E} \), \( \tilde{\mu}(a') = \mu(a') \).
PROOF: See Appendix. ♦

By PROPOSITION 10, we can restrict the issuer to doing nothing \textit{ex ante} and, after seeing her information, designing a single \textit{ex-post} security $S: \mathcal{Y} \to [0, \bar{y}]$. We will refer to this restriction of the GSD Game as the \textit{Ex-Post Security Design (EPSD) Game}.

While the EPSD game is simpler than the GSD game, the issuer’s security is still an infinite-dimensional signal of her type. The infinite-dimensional signaling problem has not been solved in general.\textsuperscript{36} However, we can solve it in an important special case, using results from prior sections. As in section 4, we will assume the issuer’s information satisfies the Hazard Rate Ordering property. Moreover, we will restrict the issuer to the set of monotone securities, which is defined as\textsuperscript{37}

$$M = \{S: \mathcal{Y} \to [0, \bar{y}]: S(y) \text{ and } y - S(y) \text{ are nonnegative and nondecreasing in } y \in \mathcal{Y},\}.$$ 

This restriction is common in the security design literature.\textsuperscript{38} One justification is that the issuer has free disposal over her cash flow $Y$ and can also borrow short-term to inflate it. Hence, if the payment to the issuer were decreasing in $Y$, the issuer would freely dispose of some cash in order to raise her payoff. And if, alternatively, the payoff to the security holders were falling in $Y$, the issuer would secretly borrow short-term to inflate $Y$, pay the security holders less, and then repay the loan. Finally, nonnegativity is motivated by the common feature of limited liability in securitization deals. [←--COMMENT: the original definition of $M$ was

\textsuperscript{36} Researchers have studied the two-dimensional signaling problem; see, e.g., Quinzii and Rochet (1985). We are not aware of solutions in three or more dimensions.

\textsuperscript{37} One can also define a corresponding notion of monotonicity in the GSD game. For any given vector $A$ of ex-ante securities, let $X_A$ denote the set $\{A(y) : y \in \mathcal{Y}\}$ of possible \textit{ex-ante} security payout vectors. Let an issuer’s securities in the GSD game be \textit{monotone} if (a) each \textit{ex-ante} security $A_i(y)$ is nondecreasing in the cash flow $y$ and (b) for each type $t$, the payout $P_j^t(x)$ of each \textit{ex-post} security $j = 1, \ldots, L_t$ is nonnegative and nondecreasing in the vector $x \in X_A$ of \textit{ex-ante} security payouts, as is the residual $y - \sum_{j=1}^{L_t} P_j^t(x)$. It should be clear that if the issuer’s securities are monotone in the GSD game, then her corresponding \textit{ex-post} security is monotone in the EPSD game.

\textsuperscript{38} See, for example, DeMarzo (2005), DeMarzo and Duffie (1999), Frankel and Jin (2015), Hart and Moore (1995), Matthews (2001), and Nachman and Noe (1994).
If \( y_0 \) were positive, I think \( S(y_0) \) could take any value in \([0, y_0]\).]

Finally, we will assume the cash flow is discrete:

**Assumption C (Discrete Cash Flow).** \( Y \in \mathcal{Y} = \{y_0, \ldots, y_n\} \).

Henceforth we normalize \( y_0 \) to zero and write \( y_n \) as \( \bar{y} \). Discreteness will be relaxed in the section 6.

Given her information \( t \), if the issuer sells security \( S \in M \) for the price \( p \), her total payoff is \( p + \delta E[Y - S(Y) | t] = \delta E[Y | t] + p - \delta E[S(Y) | t] \). As before, we ignore the exogenous discounted expected cash flow \( \delta E[Y | t] \) and simply define the issuer’s payoff to be her surplus from the sale, \( p - \delta E[S(Y) | t] \).

Let \( S_t \) denote the security designed by an issuer of type \( t \). For convenience, we now define intuitive, fairly priced equilibria in the context of EPSD games. Each definition is the natural restriction of the analogous definition for a GSD game.

**EPSD Equilibrium.** A perfect Bayesian equilibrium of the EPSD game is a security design \( S_t \in M \) and price cap \( \bar{\rho}_t \in \mathbb{R}_+ \) for each type \( t \), as well as a price function \( \hat{p}(S, \bar{\rho}) \) and belief function \( \hat{\mu}(t | S, \bar{\rho}) \), with the following properties:

4. **Payoff Maximization:** for all \( t \), the issuer’s choice \((S_t, \bar{\rho}_t)\) solves

\[
\max_{S,\bar{\rho}} \left\{ \hat{p}(S, \bar{\rho}) - \delta E[S(Y) | t] \right\} \text{ subject to } S \in M.
\]

5. **Competitive Pricing:** for any monotone security \( S \in M \) and price cap \( \bar{\rho} \), the price function \( \hat{p}(S, \bar{\rho}) \) equals \( \min \left\{ \bar{\rho}, \sum_t E[S(Y) | t] \hat{\mu}(t | S, \bar{\rho}) \right\} \).

6. **Rational Updating:** the investors’ belief function \( \hat{\mu}(t | S, \bar{\rho}) \) follows Bayes’s rule when applicable.
**Fair Pricing (EPSD Game).** An equilibrium \( \left( (S_t, \tilde{\rho}_t)_{t=0}^{T}, \hat{p}, \hat{\mu} \right) \) of the GSD game is *fairly priced* if, for each type \( t \), the price of *ex-post* security \( S_t \) equals its expected value conditional on the issuer’s type: \( \hat{p}(S_t, \tilde{\rho}_t) = E\left[ S_t(Y) \mid t \right] \).

The *outcome* of the EPSD game is the function \( \hat{u}(t) = \hat{p}(S_t, \tilde{\rho}_t) - \delta E\left[ S_t(Y) \mid t \right] \) giving the securitization payoff of each type \( t \).

**The Intuitive Criterion (EPSD Game).** A perfect Bayesian equilibrium \( \left( (S_t, \tilde{\rho}_t)_{t=0}^{T}, \hat{p}(\cdot), \hat{\mu}(\cdot \mid \cdot) \right) \) of the EPSD game, with outcome \( \hat{u}(\cdot) \), is *intuitive* if, for any security \( S \in M \) and revenue cap \( \tilde{\rho} \in \Re_+ \) for which \( \tilde{\rho} \geq \hat{u}(t) + \delta E\left[ S(Y) \mid s \right] \) for some type \( t \),

\[
\tilde{\rho} < \hat{u}(s) + \delta E\left[ S(Y) \mid s \right]
\]

implies \( \hat{u}(s \mid S, \tilde{\rho}) = 0 \).

We solve the EPSD game as follows. Consider the following set of elementary monotone securities:

\[
F_i^*(Y) = (y_i - y_{i-1}) \times 1[Y \geq y_i]
\]

for \( i = 1, \ldots, n \).

That is, elementary security \( i \) pays off if and only if \( Y \) is at least \( y_i \), in which case it pays the incremental cash flow \( y_i - y_{i-1} \). Further, let \( f^*(t) = E\left[ F^*(Y) \mid t \right] \) be the vector of expected payoffs of the \( n \) elementary securities, conditional on the issuer’s type \( t \). Letting \( a^* \) be a row vector consisting of \( n \) ones, the endowment \( (a^*, f^*) \) is then the maximal splitting of the cash flows \( Y \) into a set of monotone securities.

In particular, *any* monotone security \( S \) can be replicated by an allowable portfolio \( q^S \) from the endowment \( (a^*, f^*) \) and vice versa. Specifically, let \( q^S \) denote the row \( n \)-vector whose \( i \)th component is \( q^S_i = \left[ S(y_i) - S(y_{i-1}) \right] / (y_i - y_{i-1}) \). Clearly, \( q^S F^*(Y) = S(Y) \) and, since \( S \) is monotone, \( 0 \leq q^S \leq a^* \). Conversely, let \( S^q(y) = q^F(y) \). Then
This isomorphism allows us to use our prior results to analyze the security design equilibrium. First we extend our results from Section 2:

**PROPOSITION 11.** Suppose the issuer is restricted to monotone securities and the issuer’s information satisfies FOSD: for any cutoff \( y \) and types \( t > s \), \( \Pr(Y \leq y | s) \geq \Pr(Y \leq y | t) \). Then in any intuitive equilibrium of the Security Design Game, the issuer’s optimal security \( S^*_t(Y) \) equals \( q^*(t)F^*(Y) \) where \( q^*(t) \) solves (3).

**PROOF:** See Appendix.

The intuition for **PROPOSITION 11** is that the portfolio \( F^* \) is a maximally fine, monotone tranching of the cash flow \( Y \). Hence, the ex post design of a single monotone security derived from a random cash flow \( Y \) is equivalent to the issuer’s first tranching the cash flow as finely as possible and then, on seeing its type \( t \), choosing optimal quantities of each tranche to issue. In other words, the ex post security design game is equivalent to the asset sale game with “infinite splitting” ex ante. Hence, the payoff curves that correspond to these two cases coincide in Figure 3.

### 5.2. The Optimality of Debt

Because the securities \( F^* \) are prioritized, we can also use the results of Section 4 to develop a much stronger characterization of the optimal ex post security design. Indeed, if we strengthen our informational assumption to HRO, then the securities \( F^* \) will display increasing information sensitivity, and thus \( q^*(t) \) will have the hurdle class property from **PROPOSITION 7.** But if \( q \) has hurdle class \( c \), then the corresponding security design is

\[
S^q_t(Y) = qF^*(Y) = q^*_c(t)(y_c - y_{c-1}) \times \mathbb{1}[Y \geq y_c] + \sum_{i=1}^{c-1}(y_i - y_{i-1}) \times \mathbb{1}[Y \geq y_i] \\
= \min(Y, y_{c-1} + q^*_c(t)(y_c - y_{c-1}))
\]

A similar construction appears in Hart and Moore (1995), although they use it for a different purpose.
That is, \( S^q \) is equivalent to a standard debt contract with face value \( D = y_{c-1} + q^*_c(t)(y^*_c - y_{c-1}) \).

This observation leads to the main result of this section. Let \( v(D, t) = E[Y \wedge D \mid t] \) denote the expected payout of a standard debt contract with face value \( D \) conditional on the issuer’s type \( t \). For \( t \in \{0, \ldots, T\} \), and starting with \( D_0^* = y_n \), define the sequence of face values \( D_0^* > D_1^* > \ldots > D_T^* > 0 \) according to

\[
v(D_{i+1}^*, t + 1) - \delta v(D_{i+1}^*, t) = (1 - \delta) v(D_i^*, t).
\]

Equation (6) is the analogue, in the case of simple debt, to condition (5), which was used to characterize the equilibrium under IIS: each type \( t + 1 \) chooses the debt contract that makes the next lower type \( t \) just willing not to mimic. We then have the following characterization for the security design game:

**Proposition 12.** Suppose the issuer’s information satisfies HRO and the issuer is restricted to monotone securities. Then in the unique intuitive equilibrium of the Security Design Game, the optimal security design is standard debt, with type \( t \) issuing debt with face value \( D_t^* \); that is, \( S^*_t(Y) = \min(Y, D_t^*) \). The equilibrium face value \( D_t^* \) is positive and strictly decreasing in the issuer’s type \( t \). The equilibrium price for debt with face value \( D \) is \( v(D, \max\{t : D_t^* \geq D\}) \).

**Proof:** See Appendix.

**Proposition 12** establishes that under HRO, standard debt is the unique optimal ex post security design, in which the level of the debt (or equivalently, its seniority with respect to the underlying cash flows) signals the issuer’s information. In contrast, DeMarzo and Duffie (1999) show that if the issuer must design a single security to sell before learning its type \( t \), standard debt is also the optimal security design, but with a crucial difference: In their model the debt level is fixed ex ante and the proportion of it that the issuer retains serves as the signal of quality. Intuitively, retaining more shares lowers the payout to

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41 The left side of (6) is strictly increasing in \( D \) for \( D < y_n \). Because \( v(D, t+1) > V(D_t, t) \) and \( v(0, t) = 0 \), a unique solution to (6) exists, and \( D_{T+1}^* \in (0, D_T^*) \).
investors by the same proportion given any realization of the underlying state $Y$. Here, lowering the face value also lowers the payout by a constant proportion, but only in states in which the security is not in default. These states are more likely to occur when the issuer has high quality assets. Thus, a lower face value is more efficient than retention as a way for an issuer to signal high quality and, by PROPOSITION 3 and PROPOSITION 4, the Intuitive Criterion selects the efficient equilibrium.

Nachman and Noe (1994) also show that debt is the optimal ex post security design when an issuer must raise a fixed amount of cash in order to fund a worthwhile project. The all-or-nothing nature of the financing problem leads to a pooling equilibrium and equilibrium mispricing. Here, the issuer’s flexibility in the amount of cash raised allows for a fairly priced equilibrium in which the issuer’s information is revealed through the choice of face value.

To summarize, our results thus far show that an issuer will optimally choose to sell claims to its most senior (and thus least informationally sensitive) cash flows. If it can create any number of securities ex ante, the issuer will maximally tranche its cash flow and signal its information ex post via the most junior tranche that it chooses to sell. More precisely, it will respond to more positive information by retaining more of its junior securities, which leads investors to assign higher prices to the securities that it does sell. Finally, if the issuer can design a security after learning its type, it will use a standard debt contract in which a lower face value signals higher quality.

6. Continuous Security Design

The results of section 5 assume discrete distributions of the issuer’s type $t$ and the final asset value $Y$. In many applications it is more convenient to consider a continuous type space and assets with continuous payoffs. We develop such an extension heuristically in Section 6.1. We then show, in Section 6.2, that the discrete models converge to the continuous one. An advantage of the continuous model is that the optimal ex post security design is given by a simple differential equation.

In this section we also show another prediction of our model that has been confirmed empirically. Suppose the issuer’s final cash flow consists of repayments of a pool of
loans, of which some known proportion have been issued without requiring documentation of income or assets. Let the issuer’s type be a local macroeconomic shock, whose distribution is independent of the proportion of these no-documentation loans. Assume, moreover, that when the proportion of no-documentation loans is higher, the shock has a larger effect on the issuer’s final cash flow. In this setting, an issuer with a higher proportion of no-documentation loans will choose to design a security that has a lower face value, both conditional and unconditional on her information about the macro shock. This conforms with the finding of Begley and Purnanandam (2016) that issuers retain larger proportions of the face value of RMBS pools that contain a higher proportion of no-documentation loans.

6.1. Characterizing the Optimal Security

As before, let $v(D, t) = E[Y \wedge D | t]$ denote the payoff of simple debt with face value $D$. Letting $\Delta$ be the increment between types $t$, the incentive compatibility condition (6) becomes

$$v(D^*, t + \Delta) - \delta v(D^*, t) = (1 - \delta) v(D^*, t).$$

(7)

For a continuous type space with full support $[0, T]$, we consider the limit of this incentive constraint as $\Delta$ becomes small. Taking the derivative with respect to $\Delta$ at $\Delta = 0$ yields the following differential equation and initial value problem for the face value, which we denote $D^*_t$ to distinguish it from the discrete solution:

**Continuous Initial Value Problem (CP).**

$$\frac{\partial}{\partial t} D^*_t = -\frac{v_2(D^*_t, t)}{(1 - \delta) v_1(D^*_t, t)} - \frac{\delta E\left[\min(D^*_t, Y) | t\right]}{(1 - \delta) \Pr(Y > D^*_t | t)}$$

(8)

with $D^*_t : [0, T] \rightarrow \mathbb{R}$, together with the initial value

$$D^*_0 = \bar{y},$$

(9)

where $\bar{y} = \inf \{y : \Pr(Y \geq y) = 0\}$ is the essential supremum of $Y$.  

37
The boundary condition (9) holds as the lowest type $t = 0$ is unconstrained in (3) and thus sells $Y$ in its entirety. Equivalently, we may interpret this contract as an outright sale of the asset.

We now show that if there exists a unique solution to the above differential equation, it is a separating equilibrium of the continuous model:

**Proposition 13.** Assume HRO and the existence of a unique, continuous solution $D^\infty$ to CP. Then there is a separating equilibrium of the continuous model of the security design game in which, given its type $t$, the issuer announces a security whose payout function is $S^*(Y) = \min(D^\infty_t, Y)$.

**Proof:** See online appendix (DeMarzo, Frankel, and Jin 2015). ♦

We illustrate this result with a numerical example.

**Example 3.** Suppose the cash flows $Y$ are lognormally distributed conditional on the issuer’s information. Let the issuer’s information be the drift of $Y$ (the mean of $\ln(Y)$) so that $Y = 100e^{t - \frac{1}{2}\sigma^2 + \alpha Z}$ where $Z$ is standard normal. Suppose $t \in [0, .25]$ and $\delta = 0.95$. We derive the optimal face value of the debt as a function of $t$ for different volatilities $\sigma$ in Figure 5.
Note that the debt choice is decreasing in the issuer’s type $t$. On the other hand, the debt choice is not monotone in the (publicly known) volatility $\sigma$. But because the quality of the debt depends upon both face value and volatility, a more meaningful comparison is to compute the amount of cash raised by the issuer, $E[\min(D_t^\infty, Y) \mid t]$, for these same three cases. Recall that the issuer’s payoff equals the gains from trade $(1-\delta)E[\min(D_t^\infty, Y) \mid t]$: the difference in discount rates, $1-\delta$, multiplied by the expected portion of the final cash flow $Y$ that is transferred from the issuer to investors.
As shown in Figure 6, the amount of cash raised (and thus the issuer’s payoff) is decreasing in the volatility of the cash flows: it is easier to borrow against less risky assets. Thus, the signaling problem can induce an implied risk aversion for the issuer even though all agents in the model are risk neutral.42

6.2. Existence, Convergence, and Comparative Statics

PROPOSITION 13 assumes the existence of a unique solution to CP. In this section we provide sufficient conditions for this property to hold. Under these conditions, we show also that the solution to the discrete problem (6) converges to the continuous solution in the continuous limit. Hence, the unique solution to CP is close to the unique intuitive equilibrium in the discrete case. This result is important since there may be equilibria other than CP in the continuous case. All results in this section are proved in our online appendix (DeMarzo, Frankel, and Jin 2015).

42 On the other hand, the convexity of the curves in Figure 6 implies that the issuer can benefit from additional private information. The impact of both private information and idiosyncratic risk on the choice of assets to pool is explored in DeMarzo (2005).
In addition, a researcher may wish to embed the security design problem in a continuous model in which the joint distribution of types and shocks depends on the actions chosen by the issuer and other players in a prior “pregame” period (e.g., Frankel and Jin 2015). We show that as the gaps between types and shocks shrink to zero, the gap between profits in the discrete and continuous model also shrinks to zero, uniformly in the joint distribution of types and shocks (and thus in the pregame action profile). This result may be used in applications to show that the issuer's optimal action in the discrete case converges to that of the continuous model.43

In order to state our result, we first specify a continuous model as well as a sequence of discrete models $i = 1, 2, \ldots$ that converge to the continuous one. We first describe the continuous model, which we denote “model $\infty$”. Let the issuer's type $t \sim G$ have full support $[0,1]$ and let the final asset value $Y$ have full support $[0, \overline{y}]$ with conditional distribution $H(y|t)$ given the issuer’s type $t$. We assume $H$ is continuously differentiable, has no atoms, and satisfies the Hazard Rate Ordering property. We also assume $H$ satisfies the following technical continuity property.

**LIPSCHITZ-H (L-H).** The conditional distribution function $H$ is Lipschitz continuous and also has some minimum sensitivity to its arguments: there are constants $k_0, k_1 \in (0, \infty)$ such that for all $t$ in $[0,1]$ and $y$ in $[0, \overline{y}]$, the derivative $H_1(y|t)$ is in $(k_0, k_1)$ and $-H_2(y|t)$ lies in $[k_0, y(\overline{y} - y), k_1]$.44,45 Furthermore, both partial derivatives of $H$ are Lipschitz continuous in the type $t$: there is a constant $k_2$ in $(0, \infty)$ such that for all $y$ in $[0, \overline{y}]$ and $t', t''$ in $[0,1]$, both

$$|H_1(y|t') - H_1(y|t'')| \quad \text{and} \quad |H_2(y|t') - H_2(y|t'')|$$

are less than $k_2|t' - t''|$.46

43 The convergence of the issuer’s optimal action will generally require additional assumptions, such as the strict quasiconcavity of the issuer’s expected total profits as a function of the issuer’s pregame action.

44 A higher type $t$ is good news about the shock and thus lowers $H(y|t)$. Accordingly, we state the bounds in terms of $-H_2(y|t)$ which is nonnegative.

45 As $H(0,t)$ and $H(\overline{y}, t)$ are identically zero and one, respectively, we cannot require that $H_2(y|t)$ be sensitive to the type $t$ for all cash flows $y$. However, we can require that as $y$ moves away from zero or $\overline{y}$, this sensitivity rises at least linearly in $y$. This is ensured by the factor $y(\overline{y} - y)$ in the lower bound on $-H_2(y|t)$.44,45
We now define a sequence of discrete models \( i = 1,2,\ldots \) that converge to model \( \infty \). Let \((N_i)_{i=1}^{\infty}\) and \((N_i')_{i=1}^{\infty}\) be any two increasing sequences of positive integers. In model \( i \), let the gaps between adjacent types \( t \) and shocks \( Y \) be \( \Delta_i = 1/N_i \) and \( \Delta'_i = \bar{y}/N'_i \), respectively. That is, \( t \) lies in \( S_i = \{0,\Delta_i,\ldots,1-\Delta_i,1\} \) and \( Y \) lies in \( S'_i = \{0,\Delta'_i,\ldots,\bar{y}-\Delta'_i,\bar{y}\} \). By construction, both gaps \( \Delta_i \) and \( \Delta'_i \) converge to zero as \( i \) goes to infinity. Let the conditional distribution of \( Y \) in model \( i \) be the restriction of the continuous distribution function \( H \) to types \( t \) in \( S_i \) and payoffs \( y \) in \( S'_i \). Similarly, the distribution of \( t \) in model \( i \) is the restriction of \( G \) to types \( t \) in \( S_i \).\(^{46}\)

Let \( E^i \) and \( E^\infty \) denote the expectations operators in models \( i \) and \( \infty \), respectively. Let \( v^i(D,t) = E^i[\min(D,Y) \mid t] \) denote the expected payout of simple debt with face value \( D \) in model \( i \) given a type \( t \in S_i \). Let \( v^\infty(D,t) = E^\infty[\min(D,Y) \mid t] \) denote the expected payout of the same security in model \( \infty \) given a type \( t \in [0,1] \).

Fix equilibria in models \( i \) and \( \infty \) in which the issuer’s security is simple debt. Let \( D^i \) and \( D^\infty \) be the equilibrium face values of these securities for a given type \( t \). The equilibrium price of this security in model \( i \), denoted \( p^i(t) \), is simply the security’s expected payout \( v^i(D^i,t) \). And the issuer’s expected issuance profit, denoted \( u^i(t) \), is simply the gains from trade \( (1-\delta)v^i(D^i,t) \) as competition drives investors’ payoffs to zero. Similarly, in the continuous model the price \( p^\infty(t) \) of the security equals the expected payout \( v^\infty(D^\infty,t) \) and the issuer’s profit \( u^\infty(t) \) equals the expected gains from trade \( (1-\delta)v^\infty(D^\infty,t) \).

We now state our first result, which concerns existence and uniqueness.

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\(^{46}\) That is, in model \( i \), the probability that the type does not exceed some \( t \) in \( S_i \) is \( G(t) \), while the probability, conditional on a type \( t \), that the shock \( Y \) does not exceed some \( y \) in \( S'_i \) is \( H(y \mid t) \).

1. There exists a unique function $D^\infty$ that satisfies CP for $v = v^\infty$. This function is decreasing and differentiable in the type $t$, and takes values in $(0, \bar{y}]$. The associated price and profit functions, $p^\infty$ and $u^\infty$, are decreasing and continuous in the type $t$.

2. For each discrete model $i=1,2,...$, there exists a unique, decreasing function $D^i$ with $D^i_0 = \bar{y}$ and satisfying (7) with $v = v^i$ for all $t \in S_i$ and $\Delta = \Delta_i$.

An intuition is as follows. The Picard-Lindelöf theorem is the usual tool for proving the existence and uniqueness of the solution to a differential equation. Unfortunately, we cannot apply this theorem directly because the differential equation (8) in CP is not Lipschitz continuous in $D^\infty$: it approaches negative infinity as $D^\infty$ approaches $\bar{y}$.

We sidestep this difficulty in the following way. We define upper and lower bounds on $D^\infty$ using a modification of equation (8) that is Lipschitz continuous with constant $k$. We then show that as $k$ grows, these upper and lower bounds approach the same limit, which satisfies (8) and thus must be its unique solution $D^\infty$.

Having established the existence of a unique solution, we now show that the face value functions in the discrete model converge to the face value of the continuous model, uniformly in the issuer’s type $t$. Since the discrete equilibria uniquely satisfy the intuitive criterion, this result gives a reason to focus on the continuous equilibrium that we have identified.

PROPOSITION 15. Assume the Hazard Rate Ordering and Lipschitz-$H$. For any $\varepsilon > 0$ there is an $i^*$ such that for all models $i > i^*$ and all types $t$ in [0,1],

$$|D^i_t - D^\infty_t| < \varepsilon.$$ \(^{47}\)

\(^{47}\)Technically, the function $D^i$ is defined only for types $t$ in the discrete set $S_i$. We extend it to all types $t$ in [0,1] by evaluating it at the highest type in $S_i$ that does not exceed $t$. 43
The idea of the proof is as follows. For any model \( i = 1, 2, \ldots \) and constant \( k > 0 \), we first show that any solution \( D_i \) must lie between fixed upper and lower bounds \( \bar{D}^i_k \) and \( \underline{D}^i_k \), where these bounds are Lipschitz continuous with constant \( k \). Moreover, these bounds converge to the aforementioned upper and lower bounds on \( D^\infty \) as \( i \) grows. By the prior intuition, these bounds on \( D^\infty \) converge in turn to \( D^\infty \) as \( k \) grows. Thus, by taking \( i \) and \( k \) to infinity simultaneously, \( \bar{D}^i_k \) and \( \underline{D}^i_k \) - and thus \( D^i \) which lies between them - must converge to the unique solution \( D^\infty \) of model \( \infty \).

Begley and Purnanandam (2016) find that issuers retain larger proportions of the face value of RMBS pools that contain a higher proportion of no-documentation loans. This is predicted by our model, using the following results.

**Proposition 16.** Assume the conditional distributions \( \hat{H}(y \mid t) \) and \( \bar{H}(y \mid t) \) satisfy Hazard Rate Ordering and Lipschitz-H. Suppose also that \( v_2(D,t)/v_1(D,t) \) (which is positive) is smaller for \( H = \hat{H} \) than for \( H = \bar{H} \). Then the solutions \( \hat{D}^\infty \) and \( \bar{D}^\infty \) to CP that correspond to \( \hat{H} \) and \( \bar{H} \), respectively, satisfy \( \hat{D}^\infty_t \leq \bar{D}^\infty_t \) for any type \( t \).

**Proof:** See Appendix.

**Proposition 17.** Assume that, for any \( t \in [0, 1] \) and \( y \in [0, \bar{y}] \),

\[
\hat{H}(y \mid 1) \geq \bar{H}(y \mid 1) \quad \text{and} \quad \hat{H}_2(y \mid t) \leq \bar{H}_2(y \mid t),
\]

Then \( v_2(D,t)/v_1(D,t) \) is smaller for \( H = \hat{H} \) than for \( H = \bar{H} \).

**Proof:** Integration by parts yields

\[
v(D, t) = \int_{y=0}^{\bar{y}} \min\{y, D\} dH(y \mid t) = D - \bar{y} \int_{y=0}^{D} H(y \mid t) dy,
\]

whence \( v_2(D, t) \) equals \( -\bar{y} \int_{y=0}^{D} H_2(y \mid t) dy \) which is larger for \( H = \hat{H} \) than for \( H = \bar{H} \).

And \( v_1(y, t) \) equals \( 1 - H(y \mid t) \) which can be written as \( 1 - H(y \mid 1) + \int_{y=1}^{t} H_2(y \mid s) ds \),

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\(^{48}\) As both derivatives are negative by Lipschitz-H, this assumption means that the effect is not smaller in absolute value for the no-documentation loans.
which is smaller for $H = \bar{H}$ than for $H = \tilde{H}$. Thus, $v_2(D,t)/v_1(D,t)$ (which is positive) is larger for $\tilde{H}$ than for $\bar{H}$. 

To apply these results to the findings of Begley and Purnanandam (2016), suppose a bank plans to lend a given $\bar{y}$ dollars. Prior to lending, it flips a coin to decide the proportion of no-documentation loans to issue. After lending, it sees private information $t$ in $[0,1]$ concerning a local macroeconomic shock that affects its loans’ repayment probabilities. Assume, plausibly, that the distribution of the bank’s information $t$ does not depend on the outcome of the prior coin flip.

Let $\tilde{H}(y|t)$ and $\bar{H}(y|t)$ denote the conditional distributions of the bank’s aggregate loan repayments $Y$ conditional on the bank’s information $t$, in the low- and high-documentation cases respectively. The main effect of requiring documentation is to prevent loan applicants from exaggerating their financial strength. Thus, even if the bank receives the best possible information, $t = 1$, about the macro shock, the probability that repayments fall below any fixed threshold $y$ will be higher when there are more no-documentation loans in the pool. This is the first inequality assumed in **Proposition 17**.

Moreover, as no-documentation borrowers tend to be more financially fragile conditional on observables, the performance of their loans will be more sensitive to local macroeconomic conditions. Thus, the probability that repayments will fall below any given threshold $y$ will be more sensitive to the bank’s information $t$ about the local macro shock if the pool contains more no-documentation loans. This is captured by the second inequality assumed in **Proposition 17**.

Now combining the two preceding propositions, the model predicts that if a bank includes more no-documentation loans in its pool, the face value $D_t$ of its security will be lower for any given information $t$ and hence the retained face value $\bar{y} - D_t$ will be higher. Finally, as $t$ is independent of the documentation decision, the unconditional expectation $E(\bar{y} - D_t)$ of retained face value will also be higher for the low-documentation loan pool. Our thus model predicts that banks will retain higher
proportions of the face value of loan pools that have more no-documentation loans, as found empirically by Begley and Purnanandam (2016).

We turn now to some useful technical results. The first gives conditions under which the convergence of the discrete model to the continuous model is uniform in various parameters. This property can be useful in applications in which the security design game is preceded by some interaction in which the issuer also chooses an optimal action, as in Frankel and Jin (2015). In such settings, the result can help establish that the issuer’s optimal choices in the discrete model are well-approximated by her optimal choice in the continuous model.

Henceforth, we assume the distribution of types $G$ is continuous with a strictly positive density $g$ that satisfies the following technical condition:

**Lipschitz-$G$ (L-$G$).** There are constants $k_3, k_4 \in (0, \infty)$ such that for all types $t$ and $t'$ in $[0,1]$, $g(t) \leq k_3$ and $|g(t) - g(t')| \leq k_4 |t - t'|$.

In some applications (e.g., Frankel and Jin (2015)), the cash flow $Y$ depends both on a shock and on the actions of the issuer and others. We capture this in a simple way as follows. Let $Y$ be the product of an exogenous random variable $z \in [0,1]$ and a *cash flow parameter* $\overline{y} > 0$ that may depend on the outcome of a prior stage of play and lies in some bounded interval $[0, y]$. Henceforth, let $H$ denote the fixed distribution of the *relative cash flow* $z$, conditional on the type $t$. We also reinterpret Lipschitz-$H$ as referring to this redefined function $H$, with $y$ and $\overline{y}$ in that assumption replaced by $z$ and $1$, respectively.

We now show that the key functions of the model converge uniformly in the distributions $G$ and $H$, the type $t \in [0,1]$, and the cash flow parameter $\overline{y}$. These key functions are the face value function $D^t$, price function $p^t$, and the conditional profit function $u^t$. We also show uniform convergence of the issuer’s unconditional expected issuance profits $E u^t = E^t[u^t(t)]$ to its continuous analogue, $E u^\infty = E^\infty[u^\infty(t)]$. Finally, let $\Pi^t(t)$

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49 Uniformity in $t$ does not apply to the expected profit functions $E u$ and $E \Pi$, as they do not depend on $t$. 
denote the issuer's conditional total profits in model \( i \): the sum of issuance profits \( u'(t) \) and the conditional expected portfolio return \( E^i[Y|t] \). Let \( E\Pi^i = E^i[\Pi^i(t)] \) denote unconditional expected total profits.\(^{50}\) We show that these converge uniformly to their continuous counterparts, \( \Pi^\infty(t) = u^\infty(t) + E^\infty[Y|t] \) and \( E\Pi^\infty = E^\infty[\Pi^\infty(t)] \).

**Proposition 18.** Fix constants \( k_0, k_1, k_2, k_3, k_4, \) and \( y \), all in \((0,\infty)\). Let \( H \) be the set of conditional distribution functions \( H(z|t) \) that satisfy Hazard Rate Ordering and Lipschitz-\( H \) with constants \( k_0, k_1, \) and \( k_2 \). Let \( G \) be the set of distribution functions \( G \) that satisfy Lipschitz-\( G \) with constants \( k_3 \) and \( k_4 \). For all \( \varepsilon > 0 \) there is an \( i^* \) such that for all models \( i > i^* \), \( G \) in \( G \), \( H \) in \( H \), \( y \) in \([0,y]\), and \( t \) in \([0,1]\), \( |\omega(t) - \omega^\infty(t)| \) is less than \( \varepsilon \) for \( \omega \) equal to \( D, p, u, \) and \( \Pi \); and \( |E\omega^i - E\omega^\infty| \) is less than \( \varepsilon \) for \( \omega \) equal to \( u \) and \( \Pi \).\(^{51}\)

Finally, we show a homogeneity property that can simplify the analysis of models in which security design is embedded (e.g., Frankel and Jin (2015)).

**Corollary 19.** The face value functions \( D^i \) and \( D^\infty \), the price functions \( p^i \) and \( p^\infty \), the profit functions \( u^i, \Pi^i, u^\infty, \) and \( \Pi^\infty \), and the issuer’s expected profits \( Eu^i, E\Pi^i, Eu^\infty, \) and \( E\Pi^\infty \), defined above, are all homogeneous of degree one in the cash flow parameter \( y \).

### 7. Conclusion

In this paper we consider study the problem of an informed issuer who wishes to sell securities to raise cash. Using the Intuitive Criterion, we identify the unique equilibrium when the issuer has a fixed set of securities, and her information has a monotone impact on the expected payout of each one. Moreover, “splitting” a security into smaller tranches before becoming informed cannot lower the issuer’s payoff.

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\(^{50}\) This last quantity is especially important in applications: if there is a pregame period, the issuer will act so as to maximize the sum of \( E\Pi^i \) (perhaps multiplied by a discount factor) and any pregame payoff.

\(^{51}\) As in Proposition 14, we extend these functions to all types \( t \) in \([0,1]\) by evaluating them at the highest type in \( S_i \) that does not exceed \( t \).
We show, further, that when securities can be ordered according to their informational sensitivity, a “pecking order” result applies: the least informationally sensitive securities are sold first. When information satisfies the Hazard Rate Ordering property and securities are prioritized by seniority (which we define in a new and flexible way), information sensitivity coincides with seniority: the issuer will issue all securities whose seniority exceeds a given threshold. More optimistic issuers choose a higher threshold and thus sell fewer securities.

We also consider the optimal design of a set of securities that are secured by a common cash flow. We show, first, that the issuer optimally waits until receiving information about this cash flow, and then issues a single security. Under the additional assumption that her information satisfies the Hazard Rate Ordering property, this security is simple debt with a face value that is decreasing in her information. An equivalent strategy is to maximally tranche the cash flow \textit{ex ante} and then, \textit{ex post}, sell securities whose seniority exceeds a threshold that is increasing in the issuer’s information. In both cases, the issuer retains a larger portion of her cash flow when she is more optimistic, which Begley and Purnanandam (2016) confirm empirically.

The preceding security design results assume discrete information and shocks. By taking limits, we show that the optimal face value of the issuer’s \textit{ex post} debt security in the continuous case is given by a simple differential equation. Moreover, the issuer's expected profits in the discrete model converge uniformly to her profits in the continuous model. Our solution in the continuous case implies that issuers with worse asymmetric information problems will retain larger tranches of their cash flows, as Begley and Purnanandam (2016) also find.

8. References


9. Appendix

**Proof of Proposition 1:** First, \( u^*(0) = (1-\delta)af(0) > 0 \) by Assumption A. Next suppose \( u^*(s) \) is well-defined and strictly positive for \( s < t \). Then for type \( t \), \( q = 0 \) is strictly feasible. Since the set of feasible \( q \) is non-empty and compact, \( u^*(t) \) and \( q^*(t) \) exist. Also, since there exists \( \epsilon > 0 \) sufficiently small so that \( \epsilon a \) is feasible, \( u^*(t) \geq (1-\delta)\epsilon af(t) \geq (1-\delta)\epsilon af(0) > 0 \).

To see that \( u^*(t) \leq u^*(s) \) for \( s < t \), note that since \( f(t) \geq f(s) \) by Assumption A, if \( q \) is feasible for \( t \) then \( q \) is also feasible for \( s \). Thus, \( (1-\delta)q^*(t)f(s) \leq u^*(s) = (1-\delta)q^*(s)f(s) \). Combining this with the constraint in (3) implies
\[
q^*(t)f(t) \leq u^*(s) + \delta q^*(t)f(s) \leq u^*(s) + \delta q^*(s)f(s) = q^*(s)f(s),
\]
whence \( u^*(t) \leq u^*(s) \).

**Proof of Proposition 2:** Consider a strategy profile in which, for each type \( t \), the issuer chooses an arbitrary price cap \( \tilde{p}^*(t) \geq f(t) \) as well as a quantity \( q^*(t) \) that solves (3). Let the beliefs function \( \mu(\cdot|\cdot,\cdot) \) be given by Bayes’s rule on the equilibrium path; we specify beliefs following a deviation below. Finally, let the price function \( p(\cdot,\cdot) \) be given by (1) which ensures Competitive Pricing.
We now verify that these strategies and beliefs satisfy fair pricing and yield the outcome $u^*(\cdot)$. Let $T(q, \overline{p}) = \{t \mid (q, \overline{p}) \text{ equals } (q^*(t), \overline{p}^*(t))\}$. If $T(q, \overline{p})$ is a singleton $\{t\}$, then $p^*(q, \overline{p}) = f(t)$ by (1). Else let $s, t \in T(q, \overline{p})$ satisfy $s < t$. By definition of $T$, $q^*(s) = q^*(t) = q$. So by (3), $u^*(s) = (1-\delta)qf(s) \geq q(f(t)-\delta f(s))$ or, equivalently, $q(f(t) - f(s)) \leq 0$. Since $f(t) \geq f(s)$ by ASSUMPTION A, if $q^1 > 0$ then $f(t) = f(s)$. Since this is true for all types in $T(q, \overline{p})$, $q^1 > 0$ implies $p^*(q, \overline{p}) = f^*(t)$ for all $t \in T(q, \overline{p})$ by (1); i.e. the equilibrium is fairly priced. Thus $q^*(t) p^*(q^*(t), \overline{p}^*(t)) = q^*(t) f(t)$, and the equilibrium outcome is $u^*(t) = (1-\delta)q^*(t)f(t)$ as claimed.

It remains to verify the incentive constraints for each type. Suppose type $s$ mimics type $t$. Since the equilibrium is fairly priced, type $s$ does not gain from this deviation as long as the following inequality holds.

$$IC(s,t): \quad q^*(t) ( f(t) - \delta f(s) ) \leq u^*(s) = (1-\delta) q^*(s) f(s).$$

We must verify $IC(s,t)$ for all $s, t$. We will show that if $IC(s,s')$ holds for all $s, s' < t$ (which is trivially true when $t = 0$), then it holds for all $s, s' \leq t$. For $s' < t$, $IC(s',t)$ must hold by (3), while $IC(t,s')$ holds by the following lemma with $p = f(s')$ and $q = q^*(s')$; the lemma’s assumptions hold as $f(s') \leq f(t)$ and since $IC(s,s')$ holds for all $s < t$ by induction.

**LEMMA.** Let $p \in \mathcal{R}^n_v$ and let $q \in \mathcal{R}^n_v$ satisfy $q \leq a$. Suppose $q(p-f(t)) \leq 0$ and, for all $s < t$, $q(p-\delta f(s)) \leq u^*(s)$. Then $q(p-\delta f(t)) \leq u^*(t)$.

**PROOF OF LEMMA:** Note that $q(p-\delta f(t)) = q(p-f(t))+(1-\delta)qf(t) \leq (1-\delta)qf(t)$. Now if $q$ satisfies condition (ii) in (3) for $t$, then $(1-\delta)qf(t) \leq u^*(t)$ so we are done. Else let $\alpha$ be the largest scalar such that $\alpha q$ satisfies (ii). Then $\alpha < 1$ and, for some $s < t$, $\alpha q(f(t)-\delta f(s)) = u^*(s) \geq q(p-\delta f(s))$. Rearranging, and using the fact that $\alpha < 1$ and the definition of $u^*$, we obtain

$$q(p-\delta f(t)) \leq (1-\delta)\alpha qf(t) + \delta(1-\alpha)q(f(s)-f(t)) \leq (1-\delta)\alpha qf(t) \leq u^*(t)$$

as claimed. 

We now define investor beliefs $\mu^*$ off-equilibrium and show that no type deviates to any $(q, \overline{p})$ that lies off the equilibrium path: for which $T(q, \overline{p}) = \emptyset$. Suppose type $t'$ deviates to $(q, \overline{p})$ and receives price $p$. Such a deviation is profitable if $q(p-\delta f(t')) > u^*(t')$ or, equivalently, if $qp > u^*(t') + \delta qf(t')$. Thus, $u^*(t') + \delta qf(t')$ is the least securitization revenue that would make type $t'$ willing to deviate. This motivates the following specification of off-equilibrium beliefs:

$$\mu^*(\tau^*(q) \mid q, \overline{p}) = 1, \text{ where } \tau^*(q) = \min\{\arg\min_{t'} u^*(t') + \delta qf(t')\}. \quad (10)$$

That is, when investors observe an unexpected pair $(q, \overline{p})$, they believe the issuer’s type is the worst type of the set of types with the greatest incentive to deviate to $(q, \overline{p})$.\(^{52}\) Moreover, if type $\tau^*(q)$ does not gain from deviating to $(q, \overline{p})$ under these beliefs, then no type gains from deviating to $(q, \overline{p})$.

Fix $(q, \overline{p})$ and let $\tau^*$ denote $\tau^*(q)$. Given the beliefs $\mu^*$, the market price vector following a deviation to $q$ is $p^*(q, \overline{p}) = \overline{p} - f(\tau^*) \leq f(\tau^*)$. Thus, type $s$ does not gain by deviating to $q$ if the following holds:

$$IC(s, \tau^*, q): \quad q[f(\tau^*) - \delta f(s)] \leq u^*(s).$$

As noted, $IC(s, \tau^*, q)$ is implied by $IC(\tau^*, \tau^*, q)$ which, in turn, holds if $(1-\delta)qf(\tau^*) \leq u^*(\tau^*) = (1-\delta)q^*(\tau^*)f(\tau^*)$ or, equivalently, if $qf(\tau^*) \leq q^*(\tau^*)f(\tau^*)$. (Intuitively, since $\tau^*$ receives fair pricing whether it chooses $q$ or its equilibrium quantity $q^*(\tau^*)$, the deviation to $q$ is unprofitable for $\tau^*$ as long as it raises less cash.) To complete the proof that $(q^*, \overline{p}^*, p^*, \mu^*)$ is an equilibrium, it remains only to verify this inequality.

We will now show that $q$ is feasible in (3) for $\tau^* = \tau^*(q)$. This will imply $IC(\tau^*, \tau^*, q)$ by the optimality of $q^*(\tau^*)$. For suppose $q$ is not feasible for $\tau^* = \tau^*(q)$, and let $\alpha$ be the largest scalar such that $\alpha q$ is feasible for $\tau^*$. Then $\alpha < 1$ and by continuity $IC(s, \tau^*, \alpha q)$ must bind for some $s < \tau^*$. Therefore, using the definition of $\tau^*$,

\(^{52}\) These beliefs coincide with those used in the D1 (Divinity) refinement of Cho and Sobel (1990), which is stronger than the Intuitive Criterion. We do not require investors to have such beliefs. We use them merely to build an intuitive equilibrium whose outcome is given by (3).
\[ \alpha q[f(\tau^*)-\delta f(s)] = u^*(s) = u^*(s)+\delta qf(s)-\delta qf(s) > u^*(\tau^*)+\delta qf(\tau^*)-\delta qf(s). \]

(The inequality is strict by equation (10).) This can be rearranged to yield

\[ u^*(\tau^*) < (1-\delta)\alpha qf(\tau^*) + \delta q(1-\alpha)(f(s)-f(\tau^*)) \leq (1-\delta)\alpha qf(\tau^*), \]

where the last inequality follows since \( \alpha < 1 \) and \( f(s) \leq f(\tau^*) \). But this implies that \( \alpha q \) is feasible and strictly better than \( q^*(\tau^*) \) is for \( \tau^*(q) \) in (3) – a contradiction. We have verified that \( IC(\tau^*(q),\tau^*(q),q) \) always holds: \( (q^*,\bar{p}^*,p^*,\mu^*) \) is an equilibrium as claimed.

Finally, we show that \( p^* \) is nonincreasing in \( q \), whether or not \( q \) is chosen in equilibrium. We first require the following lemma, which clarifies the difference between on- and off-equilibrium beliefs.

**Lemma.** For any quantity vector \( q \), whether or not it is expected in equilibrium, beliefs are concentrated on the set \( \arg\min_{t'}[u^*(t')+\delta qf(t')] \). If \( q \) is unexpected, beliefs are concentrated on the lowest element of this set. Else, for \( q \neq a \) it is not the lowest element; generically, it is the second lowest.

**Proof of Lemma:** Say \( q \) is chosen by some type \( t \) in equilibrium. One requirement of equilibrium is that no type \( s \neq t \) strictly prefers to deviate to \( q \). Hence, the IC constraint (3) must hold for every other type \( s \neq t \), not only types \( s < t \). So for any \( s \neq t \), \( u^*(s)+\delta qf(s) \) is no less than \( qf(t) \) which, in turn, equals \( u^*(t)+\delta qf(t) \). Thus, \( t \) lies in \( \arg\min_{t'}[u^*(t')+\delta qf(t')] \). If \( q = a \), then \( t \) equals zero which, as the lowest type, must equal \( \min\{\arg\min_{t'}[u^*(t')+\delta qf(t')]\} \). But if \( q \neq a \) then (3) will bind for some types \( s < t \), each of which must thus lie in \( \arg\min_{t'}[u^*(t')+\delta qf(t')] \): \( t \) is not the lowest type in this set. Generically, (3) will bind for a single \( s < t \), whence \( t \) is the second-lowest type in the set.

Now consider two quantity vectors \( q' > q \). Let \( \tau \) be in the set of beliefs following \( q \) and \( \tau' \) be in the set of beliefs following \( q' \). Then by the preceding Lemma,

\[ \tau \in \arg\min_{t}[u^*(t)+\delta qf(t)] \quad \text{and} \quad \tau' \in \arg\min_{t}[u^*(t)+\delta q'f(t)]. \]

It follows that

\[ u^*(\tau)+\delta qf(\tau) \leq u^*(\tau') + \delta qf(\tau') \quad \text{and} \quad u^*(\tau) + \delta q'f(\tau) \geq u^*(\tau') + \delta q'f(\tau'); \]

then
subtracting the first inequality from the second and simplifying we obtain
\((q' - q)[f'(\tau) - f'(\tau')] \geq 0\) which implies \(\tau \geq \tau'\) by ASSUMPTION A.  

**Proof of Proposition 4:** By Proposition 2, \((q^*, \widetilde{p}^*, p^*, \mu^*)\) is an equilibrium. Consider any pair \((q, \widetilde{p})\) such that for some types \(t\) and \(s\),

\[ u^*(t) + \delta qf(t) \leq q\widetilde{p} < u^*(s) + \delta qf(s). \]

Then \(s \neq \tau^*(q)\) by (4), which implies \(\mu^*(s \mid q, \widetilde{p}) = 0\). Thus, \((q^*, \widetilde{p}^*, p^*, \mu^*)\) is intuitive.

Next, consider any intuitive equilibrium with outcome \(u\). First we show that it is fairly priced. Suppose that with positive probability, type \(t\) makes asset sale decision \((q, \widetilde{p})\). Assume investors respond with price vector \(p\). First say \(qp < qf(t)\): the issue is underpriced. Let \(s\) be the largest type such that \(qf(s) < qf(t)\). Then define \(\lambda \in [0,1)\) such that

\[ \lambda q[f(t) - \delta f(t)] < q[\lambda q[f(t') - \delta f(t')] \leq u(s'), \]

where the last inequality follows from the incentive constraint for type \(s'\). Thus, no type \(s' \leq s\) has an incentive to deviate to \((\lambda q, f(t))\).

On the other hand, from (11),

\[ \lambda q[f(t) - \delta f(t)] > q[p - \delta f(t)] = u(t), \]

so type \(t\) could gain from the deviation if the realized price \(p\) equals \(f(t)\). But then the Intuitive Criterion implies that

\[ 53 \text{ If there is no such type } s, \text{ then for all types } t', f(t') \geq f(t) \text{ whenever } q_i > 0. \text{ Thus, if type } i \text{ deviates to } (q, f(t)), \text{ the price will be at least } f(t), \text{ which is better for } t \text{ than } p \text{ is.} \]
\[ \mu(s' \mid \lambda, q, f(t)) = 0 \text{ for all } s' \leq s. \]

That is, investor beliefs must put weight only on types \( t' \) such that \( q \ f(t') \geq q \ f(t) \). By ASSUMPTION A, for these types \( f(t') \geq f(t) \) for all securities \( i \) such that \( q_i > 0 \). Hence, for each such security \( i \),

\[ p_i(\lambda, q, f(t)) = f(t) \wedge \sum_{t'} f_i(t') \mu(t' \mid \lambda, q, f(t)) = f(t). \]

Accordingly, type \( t \) gains from the deviation to \((\lambda, q, f(t))\) - a contradiction. Thus, it must be the case that \( q_p \geq q_f(t) \) for all types \( t \) that make sale decision \((q, \overline{p})\) in equilibrium; that is, there is no equilibrium underpricing. But from (1),

\[ q_p = q \left[ \overline{p} \wedge \sum_f f(t) \mu(t \mid q, \overline{p}) \right] \leq \sum_f q_f(t) \mu(t \mid q, \overline{p}). \]

That is, \( q_p \) is bounded above by a convex combination of \( q_f(t) \) for all types \( t \) that make sale decision \((q, \overline{p})\) in equilibrium. But as just shown, \( q_p \) is also bounded below by \( q_f(t) \) for any such \( t \). Thus, \( q_p \) must equal \( q_f(t) \) for all types \( t \) that ever choose \((q, \overline{p})\): without underpricing there can be no overpricing.

Further, if \( t, t' \) make sale decision \((q, \overline{p})\) in equilibrium and \( t > t' \), then \( f(t) \geq f(t') \) by ASSUMPTION A. Since \( q \ f(t) = q \ f(t') \), it must be that \( f(t) = f(t') \) if \( q_i > 0 \). Hence, \( p_i(q, \overline{p}) = f_i(t) \) if \( q_i > 0 \), whence the equilibrium is fairly priced.

It remains to show that in any intuitive equilibrium, the outcome \( u \) equals \( u^* \) and the equilibrium asset sale function \( q(\cdot) \) solves (3).

Let \( q^*(\cdot) \) be any solution to (3). By PROPOSITION 3, \( u \leq u^* \). Let \( t \) be the smallest type such that either (a) \( u(t) < u^*(t) \) or (b) \( q(t) \) does not solve (3). As \( u(s) = u^*(s) \) for all \( s < t \), the payoff \( u(t) \) equals its maximum value of \( u^*(t) \) if and only if \( q(t) \) solves (3).

Hence, conditions (a) and (b) are equivalent: they both must hold for type \( t \). Since \( u(t) < u^*(t) \), it follows that \( u(t) + \delta q^*(t)f(t) \) is less than \( u^*(t) + \delta q^*(t)f(t) \), which equals \( q^*(t)f(t) \) by (3). Also by (3), for all \( s < t \), \( q^*(t)f(t) \) does not exceed \( u^*(s) + \delta q^*(s)f(s) \), which
equals $u(s) + \delta q^*(t)f(s)$ by hypothesis. Hence, there exists a maximum price vector $\overline{p}$ close to but less than $f(t)$ such that for any type $s < t$,

$$u(t) + \delta q^*(t)f(t) < q^*(t)\overline{p} < q^*(t)f(t) \leq u(s) + \delta q^*(t)f(s).$$

But then intuitive beliefs put no weight on types $s < t$ if $(q^*(t), \overline{p})$ is observed. Hence, $p(q^*(t), \overline{p}) = \overline{p}$. But then because $u(t) < q^*(t)(\overline{p} - \delta f(t))$, type $t$ could gain by deviating to $(q^*(t), \overline{p})$ – a contradiction. It follows that there is no such minimum type $t$: for all types $t$, $u(t) = u^*(t)$ and $q^*(t)$ solves (3). ♦

**Proof of Proposition 8:** First, $q^*(0) = a$ by (3) and Assumption B. For $t > 0$, since $f(t) - \delta f(t-1) > 0$ by Assumption A and Assumption B, equation (5) has a unique solution in C. Equation (5) is simply the incentive constraint in (3) for $s = t-1$. It remains to show that this constraint binds when $s = t-1$. Suppose instead that

$$q^*(t)[f(t) - \delta f(t-1)] < u^*(t-1) = (1-\delta)q^*(t-1)f(t-1).$$

From the definition of $q^*(t-1)$ in (3), for any $s < t-1$,

$$q^*(t-1)(f(t-1) - \delta f(s)) \leq u^*(s).$$

Combining these two yields:

$$q^*(t)[f(t) - \delta f(t-1)] - (1-\delta)q^*(t-1)f(t-1) + q^*(t-1)(f(t-1) - \delta f(s)) < u^*(s)$$

or equivalently,

$$q^*(t)(f(t) - \delta f(s)) + \delta (q^*(t-1) - q^*(t)) (f(t-1) - f(s)) < u^*(s).$$

From Proposition 7, $q^*(t-1) \geq q^*(t)$, and from Assumption A, $f(t-1) \geq f(s)$. Thus,

$$q^*(t)(f(t) - \delta f(s)) < u^*(s),$$

which implies that none of the incentive constraints in (3) bind for $q^*(t)$. This contradicts the optimality of $q^*(t)$ unless $q^*(t)f(t) = a f(t)$. But by the initial supposition,

$$q^*(t)f(t) < u^*(t-1) + \delta q^*(t)f(t-1) \leq a f(t-1) \leq a f(t).$$

Hence, the incentive constraint for $t-1$ must bind. ♦
**Proof of Proposition 9:** The proposition follows from Proposition 7 as long as we can show that if \( j \) is junior to \( i \) then it is more informationally sensitive; that is, for \( t > s \),

\[
\frac{f_j(t)}{f_j(s)} > \frac{f_i(t)}{f_i(s)}.
\]

Fix types \( t > s \). As both informational sensitivity and priority are invariant to an arbitrary rescaling of each security’s payoff, w.l.o.g. we can let \( f_j(t) = f_i(t) = 1 \). Hence, we need to show that for \( s < t \), \( f_j(s) < f_i(s) \).

Because securities are prioritized, \( F_j(Y) = h(Y)F_i(Y) \) for some nondecreasing function \( h \) which is not degenerate – that is, which is not almost surely a constant on the set \( F_i(Y) > 0 \). From our earlier normalization,

\[
0 = f_j(t) - f_i(t) = E[F_j(Y_i) - F_i(Y_i)] = E[(h(Y_i) - 1)F_i(Y_i)]
\]

\[
= E[(h(Y_i) - 1)F_i(Y_i) | F_i(Y_i) > 0].
\]

Thus, because \( h \) is nondegenerate on the set \( F_i(Y_i) > 0 \), it must be strictly below 1 and strictly above 1 with positive probability on this set.

Next we make use of the following lemma.

**Lemma:** Suppose \( Y \) satisfies HRO. Then there exists a random variable \( Z \) with the same support as, and independent of, \( Y \), such that \( Y_s \) and \( Y \wedge Z \) have the same distributions.

**Proof:** Let \( \bar{y} \) denote the essential supremum of \( Y \): \( \bar{y} = \inf \{ y : \Pr(Y \geq y) = 0 \} \).

Fix \( s < t \), and let \( R(y) = \frac{\Pr(Y_i \geq y)}{\Pr(Y_i \geq y)} \) for \( y < \bar{y} \) and zero for \( y > \bar{y} \), with \( R(\bar{y}) \) defined so that \( R \) is left-continuous. Note that \( R(0) = 1 \), and \( R \) is strictly decreasing on the support of \( Y \) and constant elsewhere. Thus we can define a new, independent random variable \( Z \) with distribution \( \Pr(Z \geq y) = R(y) \) and note that \( Z \) has the same support as \( Y \).\(^{54}\) Finally,

\(^{54}\) See e.g. Theorem 12.4 of Billingsley (1986).
\[ \Pr(Y_t \land Z \geq y) = \Pr(Y_t \geq y, Z \geq y) = \Pr(Y_t \geq y) \Pr(Z \geq y) = \Pr(Y_t \geq y) \frac{\Pr(Y_t \geq y)}{\Pr(Y_t \geq y)} = \Pr(Y_t \geq y), \]

as claimed. \( \blacksquare \)

Using the lemma, together with the monotonicity of the securities, we have

\[ f_i(s) = E[F_i(Y) \mid s] = E[F_i(Y)] = E[F_i(Y_t \land Z)], \]

and similarly for \( f_j \). If \( h(z) < 1 \), then

\[ E[F_j(Y_t \land z)] = E[h(Y_t \land z)F_j(Y_t \land z)] \leq E[F_i(Y_t \land z)], \]

where the inequality is strict unless \( F_i(z) = 0 \). Conversely, if \( h(z) \geq 1 \) then

\[
E[F_j(X_t \land z)] = E[F_j(X_t) - (F_j(X_t) - F_j(z))^+] \\
= 1 - E[(h(X_t)F_j(X_t) - h(z)F_j(z))^+] \\
\leq 1 - E[(F_j(X_t) - F_j(z))^+] = E[F_j(X_t \land z)].
\]

Thus, \( f_j(s) = E[F_j(Y_t \land Z)] < E[F_i(Y_t \land Z)] = f_i(s) \) where the inequality is strict because there is a positive probability that \( F_i(Z) > 0 \) and \( h(Z) < 1 \). \( \blacksquare \)

**Proof of Proposition 10.** We first verify that the profile \( \bar{E} \) is a fairly priced equilibrium. Competitive Pricing and Rational Updating hold by Pricing and Beliefs. Fair Pricing follows from Rational Updating as long as investors can infer the expected payout \( E[\bar{P}_t^i(Y) \mid t] \) of type \( t \)'s security by observing the security payout function \( \tilde{P}_t^i \). If they cannot, there must be two types \( t', t \) that issue the same *ex-post* security \( \tilde{P}_t^i \) such that \( E[\tilde{P}_t^i(Y) \mid t'] > E[\tilde{P}_t^i(Y) \mid t] \). But then since \( E \) is fairly priced, type \( t \) could imitate type \( t' \) in \( E \) and obtain higher securitization revenue \( E[\tilde{P}_t^i(Y) \mid t'] \) without affecting her discounted expected payout \( \delta E[\tilde{P}_t^i(Y) \mid t] \); she would deviate in \( E \). As \( E \) is an equilibrium, this cannot be true: \( \bar{E} \) must be fairly priced.
By construction, the aggregate conditional expected payout of the *ex-post* securities of each type $t$ are the same in $\tilde{E}$ as in $E$. Hence, fair pricing implies that type $t$’s securitization revenue is also the same in $\tilde{E}$ as in $E$. Accordingly, $\tilde{E}$ and $E$ have the same outcome $u(\cdot)$ as claimed. This also implies that no type wants to imitate any other type in $\tilde{E}$ (since no type wants to do so in $E$). Moreover, no type $t$ strictly prefers to choose an action $a’ = (A’, P’, \rho’)$ that is never chosen in $\tilde{E}$. Why? Since $E$ is an equilibrium, $t$’s payoff in $E$ from choosing $a’$ in $E$ cannot exceed her equilibrium payoff $u(t)$ in $E$. But beliefs following deviations in $\tilde{E}$ coincide with beliefs in $E$. So since both $\tilde{E}$ and $E$ satisfy Competitive Pricing, $a’$ must yield the same securitization revenue in $\tilde{E}$ as in $E$ – and thus the same payoff. Thus, type $t$ is willing not to choose $a’$ in $\tilde{E}$ as well: Payoff Maximization holds.

Now suppose $E$ satisfies the Intuitive Criterion. Consider any action $a’ = (A’, P’, \rho’)$ for which there is a type $t$ satisfying $\sum_j \rho^{ij} \geq u(t) + \delta E\left[W_{P'}^{A'}(Y)|t\right]$, and let type $s$ satisfy $\sum_j \rho^{sj} < u(s) + \delta E\left[W_{P'}^{A'}(Y)|s\right]$. Since $E$ is intuitive, $\mu(s|a’) = 0$. By Beliefs, then, $\hat{\mu}(s|a’) = 0$ if $a’$ is unexpected in $\tilde{E}$. And if $a’$ is sometimes chosen in $\tilde{E}$, it cannot be chosen by type $s$ as her equilibrium payoff $u(s)$ in $\tilde{E}$ exceeds the maximum payoff $\sum_j \rho^{ij} - \delta E\left[W_{P'}^{A'}(Y)|s\right]$ that she can get from $a’$ in $\tilde{E}$. Hence, $\tilde{E}$ is intuitive as claimed. •

**PROOF OF PROPOSITION 11:** It is easy to see that $S_i^*$ as defined is an intuitive equilibrium. By our previous argument, $S_i^*$ is monotone and thus feasible, and obviously provides the same equilibrium payoff $u^*(t)$. A deviation to $S_i^*$ is not profitable because the deviation from $q^*(t)$ to $q^*(s)$ is not profitable. For a deviation off-equilibrium to some monotone $S'$, consider the corresponding deviation to $q'$ where $q' F^* = S'$. Because $\arg\min_{t} \sum_{t} u^*(t) + \delta q' f(t) = \arg\min_{t} u^*(t) + \delta E[S'|t]$, the equilibrium is supported by same (intuitive) beliefs as in (4). Finally, we can establish uniqueness of the intuitive
equilibrium in the security design game in exactly the same manner as Proposition 4. Indeed, the only difference between the games is that in the asset sale game a maximum price can be set for each individual security, whereas in the security design game there is a single maximum price for the aggregate security. None of the arguments in the proof of Proposition 4, however, make use of this additional flexibility. •

Proof of Proposition 12: From Proposition 11, $S_t^*(Y) = q^*(t) F^*(Y)$. By Proposition 9, IIS holds so that from Proposition 7, $q_i^*(t) = 1$ for $i < h(t)$ and $q_i^*(t) = 0$ for $i > h(t)$. Therefore $S_i^*(y_i) = q^*(t) F^*(y_i) = y_i$ for $i < h(t)$ and

$$S_i^*(y_i) = q^*(t) F^*(y_i) = y_{h(t)-1} + q_{h(t)}^*(t) \left(y_{h(t)} - y_{h(t)-1}\right)$$

for $i \geq h(t)$. Thus, $S_t^*(Y) = \min \left(D_t^* \cdot Y\right)$ where $D_t^* = y_{h(t)-1} + q_{h(t)}^*(t) \left(y_{h(t)} - y_{h(t)-1}\right)$.

Finally, since $q^*$ and $h$ are decreasing in $t$ by Proposition 7, $D_t^*$ is decreasing in $t$. That $D_t^*$ satisfies (6) follows from Proposition 8. Finally, as the face value function is strictly decreasing, the equilibrium is fully revealing; hence, $u^*(t) = (1 - \delta)v(D_t,t)$. Off-equilibrium, suppose debt $D \in (D_{t+1}^*, D_t^*)$ is issued. The price $p$ that would make type $t$ willing to deviate satisfies

$$p = u^*(t) + \delta v(D_t,t) = (1 - \delta)v(D_t^*,t) + \delta v(D,t).$$

For any type $s < t$, the incentive constraint for $s$ implies

$$v(D_t^*,t) - \delta v(D_t^*,s) \leq u^*(s).$$

Combining these, we have

$$p + \delta \left[v(D_t^*,t) - v(D,t) - (v(D_t^*,s) - v(D,s))\right] \geq u^*(s) + \delta v(D,s),$$

where the terms in brackets is positive: the value of junior debt with face value $D_t^* - D$ is higher for type $t$ than for type $s < t$. Therefore, types $s < t$ would not find it profitable to
deviate given price \( p \). Thus, the beliefs specified in (4) must concentrate on \( t \), and so the market price given deviation to \( D \) is \( v(D,t) \).